Examples Matrices and Systems of Linear Equations

Based on power point presentations by Pearson Education, Inc. Revised by Ingrid Stewart, Ph.D.

Learning Objectives

- 1. Define a matrix (plural: matrices).
- 2. Perform allowable matrix row operations.
- 3. Define augmented matrices.
- 4. Solve systems of linear equations using the matrix method.

Example 1: Perform Matrix Operations

Perform the indicated matrix operation:

$$\begin{bmatrix} 5 & 4 \\ -3 & 7 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 8 \\ 6 & 0 \\ -5 & 3 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 5 & 4 \\ -3 & 7 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 8 \\ 6 & 0 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 5 - (-4) & 4 - 8 \\ -3 - 6 & 7 - 0 \\ 0 - (-5) & 1 - 3 \end{bmatrix} = \begin{bmatrix} 9 & -4 \\ -9 & 7 \\ 5 & -2 \end{bmatrix}$$

Example 2: Perform Matrix Operations

Perform the indicated matrix operation:

Solution:

$$5\begin{bmatrix} 5 & 4 \\ 3 & 7 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5(5) & 5(4) \\ 5(3) & 5(7) \\ 5(0) & 5(1) \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 15 & 35 \\ 0 & 5 \end{bmatrix}$$

Example 3: Perform Matrix Operations

Given the following two matrices, perform 2A - 4B.

$$\mathbf{A} = \begin{bmatrix} 5 & 4 \\ -3 & 7 \\ 0 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} -4 & 8 \\ 6 & 0 \\ -5 & 3 \end{bmatrix}$$

Solution:

$$2A = 2\begin{bmatrix} 5 & 4 \\ 3 & 7 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 6 & 14 \\ 0 & 2 \end{bmatrix}$$
then
$$2A - 4B = \begin{bmatrix} 10 & 8 \\ 6 & 14 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 16 & -32 \\ -24 & 0 \\ 20 & -12 \end{bmatrix} = \begin{bmatrix} 26 & -24 \\ -18 & 14 \\ 20 & -10 \end{bmatrix}$$

$$-4B = -4\begin{bmatrix} -4 & 8 \\ 6 & 0 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 16 & -32 \\ -24 & 0 \\ 20 & -12 \end{bmatrix}$$

Example 4: Use Gauss-Jordan Elimination (1 of 6)

Solve the following system of two linear equations in two variables using Gauss-Jordan elimination:

$$\begin{cases} -x + 4y = 5 \\ x - y = 1 \end{cases}$$

First, we will change to augmented matrix form as follows:

$$\begin{bmatrix} -1 & 4 & 5 \\ 1 & -1 & 1 \end{bmatrix}$$

The next step is to change the augmented matrix to row-echelon form using allowable row operations.

Example 4: Use Gauss-Jordan Elimination (2 of 6)

The first thing we must do is produce the number 1 in the first element in Row 1.

We will first copy Row 2. Then we will replace each element in Row 1 with the following calculation: $-1R_1$

$$\begin{bmatrix} -1 & 4 & 5 \\ 1 & -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -4 & -5 \\ 1 & -1 & 1 \end{bmatrix}$$

Example 4: Use Gauss-Jordan Elimination (3 of 6)

Next, we want the number 0 for the first element in Row 2.

We will first copy Row 1. Then we replace each element in Row 2 with the following calculation: $-1R_1 + R_2$

$$\begin{bmatrix} 1 & -4 & | & -5 \\ 1 & -1 & | & 1 \end{bmatrix} - 1R_1 + R_2 = \begin{bmatrix} 1 & -4 & | & -5 \\ 0 & 3 & | & 6 \end{bmatrix}$$

We write the instructions next to the row we want change. The R_2 in $-R_1-R_2$ tells us that we will change Row 2. We will first copy Row 1. Following are the calculations for the new Row 2:

first term: -1(1) + 1 = 0 second term: -1(-4) + (-1) = 3

constant: -1(-5) + 1 = 6

Example 4: Use Gauss-Jordan Elimination (4 of 6)

Working with the changed matrix, we now want the number 1 for the second element in Row 2.

We will first copy Row 1. Then we replace each element in Row 2 with the following calculation: $\frac{1}{3}R_2$

We write the instructions next to Row 2.

$$\begin{bmatrix} 1 & -4 & | & -5 \\ 0 & 3 & | & 6 \end{bmatrix} \frac{1}{3}R_2 = \begin{bmatrix} 1 & -4 & | & -5 \\ 0 & 1 & | & 2 \end{bmatrix}$$

Example 4: Use Gauss-Jordan Elimination (5 of 6)

Working with the changed matrix, we now want the number 0 for the second element in Row 1.

We will first copy Row 2. Then we replace each element in Row 1 with the following calculation: $4R_2 + R_1$

$$\begin{bmatrix} 1 & -4 & | -5 \\ 0 & 1 & | 2 \end{bmatrix} 4R_2 + R_1 = \begin{bmatrix} 1 & 0 & | 3 \\ 0 & 1 & | 2 \end{bmatrix}$$

We calculated the terms of the new Row 1 as follows:

first term: 4(0) + 1 = 1 second term: 4(1) + (-4) = 0

constant: 4(2) + (-5) = 3

Example 4: Use Gauss-Jordan Elimination (6 of 6)

The original augmented matrix is now in row-echelon form

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

It translates to 1x + 0y = 3 and 0x + 1y = 2.

We can therefore state that x = 3 and y = 2. Graphically this means that the two lines, defined by the two equations in the system, intersect at the point created by the ordered pair (3, 2).

Example 5: Use Gauss-Jordan Elimination (1 of 11)

Solve the following system of three linear equations in three variables using Gauss-Jordan elimination:

$$\begin{cases}
-2x - 4y - 2z = -18 \\
-4x - y + 2z = 10 \\
4x + 3y + 2z = 10
\end{cases}$$

First, we will change to augmented matrix form:

$$\begin{bmatrix} -2 & -4 & -2 & | -18 \\ -4 & -1 & 2 & | 10 \\ 4 & 3 & 2 & | 10 \end{bmatrix}$$

Note that we only used the coefficients of the variables with the coefficients for x appearing in column 1, the coefficients for y in column 2, the coefficients for z in column 3, and the constants in column 4!

Example 5: Use Gauss-Jordan Elimination (2 of 11)

The next step is to change the augmented matrix to row-echelon form using allowable row operations.

Please note that we also could have used the Addition Method to solve this system.

Example 5: Use Gauss-Jordan Elimination (3 of 11)

The first thing we must do is change the first element in Row 1 to the number 1. It is easy to achieve 1's because all we have to do is divide all elements of a row by the appropriate number.

We will first copy Rows 2 and 3. Then we replace each element in Row 1 with the following calculation: $-\frac{1}{2} \mathbf{R}_1$

NOTE: To not get confused, it is best to write the instructions next to the row you want to change!

$$\begin{bmatrix} -2 & -4 & -2 & | & -18 \\ -4 & -1 & 2 & | & 10 \\ 4 & 3 & 2 & | & 10 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & | & 9 \\ -4 & -1 & 2 & | & 10 \\ 4 & 3 & 2 & | & 10 \end{bmatrix}$$

Example 5: Use Gauss-Jordan Elimination (4 of 11)

The next thing we want is change the first element in Row 2 to the number 0. This is not so easy to achieve. To get 0's we must always add two rows R (sometimes using multiplication) to replace the row in which we want the 0.

We will first copy Rows 1 and 3. Then, we replace each element in Row 2 with the following calculation: ${}^{4R_{f}} + {}^{4R_{2}}$

$$\begin{bmatrix} 1 & 2 & 1 & | & 9 \\ -4 & -1 & 2 & | & 10 \\ 4 & 3 & 2 & | & 10 \end{bmatrix} 4R_1 + R_2 = \begin{bmatrix} 1 & 2 & 1 & | & 9 \\ 0 & 7 & 6 & | & 46 \\ 4 & 3 & 2 & | & 10 \end{bmatrix}$$

Example 5: Use Gauss-Jordan Elimination (5 of 11)

Working with the changed matrix, we want to change the first element in Row 5 to the number 0.

We will first copy Rows 1 and 2. Then we replace each element in Row 3 with the following calculation: $-4R_1 + R_3$

$$\begin{bmatrix} 1 & 2 & 1 & | & 9 \\ 0 & 7 & 6 & | & 46 \\ 4 & 3 & 2 & | & 10 \end{bmatrix} - 4R_1 + R_3 = \begin{bmatrix} 1 & 2 & 1 & | & 9 \\ 0 & 7 & 6 & | & 46 \\ 0 & -5 & -2 & | & -26 \end{bmatrix}$$

Example 5: Use Gauss-Jordan Elimination (6 of 11)

Working with the changed matrix, we want to change the second element in Row 2 to the number 1.

We will first copy Rows 1 and 3. Then we replace each element in Row 2 with the following calculation: $\frac{1}{7} \mathbf{R}_2$

$$\begin{bmatrix} 1 & 2 & 1 & 9 \\ 0 & 7 & 6 & 46 \\ 0 & -5 & -2 & -26 \end{bmatrix} \stackrel{?}{7}R_2 = \begin{bmatrix} 1 & 2 & 1 & 9 \\ 0 & 1 & \frac{6}{7} & \frac{46}{7} \\ 0 & -5 & -2 & -26 \end{bmatrix}$$

We ended up with some "nasty" fractions in Row 2!

Example 5: Use Gauss-Jordan Elimination (7 of 11)

Working with the changed matrix, we want to change the second element in Row 3 to the number 0.

We will first copy Rows 1 and 2. Then we replace each element in Row 3 with the following calculation: $5R_2 + R_3$

$$\begin{bmatrix} 1 & 2 & 1 & 9 \\ 0 & 1 & \frac{6}{7} & \frac{46}{7} \\ 0 & 5 & -2 & -26 \end{bmatrix} 5R_2 + R_3 = \begin{bmatrix} 1 & 2 & 1 & 9 \\ 0 & 1 & \frac{6}{7} & \frac{46}{7} \\ 0 & 0 & \frac{16}{7} & \frac{48}{7} \end{bmatrix}$$

We calculated the terms of the new Row 3 as follows:

first term: 5(0) + 0 = 0 second term: 5(1) + (-5) = 0

third term: $5\left(\frac{6}{7}\right) + (-2) = \frac{16}{7}$ constant: $5\left(\frac{46}{7}\right) + (-26) = \frac{48}{7}$

Example 5: Use Gauss-Jordan Elimination (8 of 11)

Working with the changed matrix, we now want to change the third element in Row 3 to the number 1.

We will first copy Rows 1 and 2. Then we replace each element in Row 3 with the following calculation5: $\frac{7}{16}$ \mathbb{R}_3

$$\begin{bmatrix} 1 & 2 & 1 & 9 \\ 0 & 1 & \frac{6}{7} & \frac{46}{7} \\ 0 & 0 & \frac{16}{7} & \frac{48}{7} \end{bmatrix}_{\frac{7}{16}}^{\frac{7}{16}} R_3 = \begin{bmatrix} 1 & 2 & 1 & 9 \\ 0 & 1 & \frac{6}{7} & \frac{46}{7} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Example 5: Use Gauss-Jordan Elimination (9 of 11)

Working with the changed matrix, we now want to change the third element in Row 2 to the number 0.

We will first copy Rows 1 and 3. Then we replace each element in Row 2 with the following calculation: $-\frac{6}{7}\mathbf{R}_3 + \mathbf{R}_2$

$$\begin{bmatrix} 1 & 2 & 1 & 9 \\ 0 & 1 & \frac{6}{7} & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix} - \frac{6}{7}R_3 + R_2 = \begin{bmatrix} 1 & 2 & 1 & 9 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Example 5: Use Gauss-Jordan Elimination (10 of 11)

Working with the changed matrix, we now want to change the third element in Row 1 to the number 0.

We will first copy Rows 2 and 3. Then we replace each element in Row 1 with the following calculation: $-1R_3 + R_4$

$$\begin{bmatrix} 1 & 2 & 1 & 9 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} - 1R_3 + R_7 = \begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Working with the changed matrix, we now want to change the second element in Row 1 to the number 0.

We will copy Rows 2 and 3. The we replace each element in Row 1 with the following calculation: $-2R_2 + R_3$

Example 5: Use Gauss-Jordan Elimination (11 of 11)

$$\begin{bmatrix} 1 & 2 & 0 & | & 6 \\ 0 & 1 & 0 & | & 4 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}^{-2R_2 + R_1} = \begin{bmatrix} 1 & 0 & 0 & | & -2 \\ 0 & 1 & 0 & | & 4 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$$
 The matrix is now in row-echelon form. It is the solution matrix.

It translates to 1x + 0y + 0z = -2 and 0x + 1y + 0z = 4 and 0x + 0y + 1z = 3.

We can therefore state that x = -2 and y = 5 and z = 3. Graphically this means that the three planes, defined by the three equations in the system, intersect at the point created by the ordered triple (-2, 4, 3).

Example 6: Interpret a Solution Matrix

Discuss the following solution matrix of a system of three linear equations in three variables x, y, and z.

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This translates to 1x + 0y + 0z = -2 and 0x + 1y + 0z = 4 and 0x + 0y + 1z = 3.

This matrix is in proper row-echelon form. The solutions for x, y, and z are

$$x = -2$$

$$y = 4$$

$$z = 3$$

Graphically, three planes intersect in exactly one point, namely (-2, 4, 5).

Example 7: Interpret a Solution Matrix

Discuss the following solution matrix of a system of three linear equations in three variables x, y, and z.

$$\begin{bmatrix} 1 & -3 & 1 & 1 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

This translates to 1x - 3y + 1z = 1 and 0x + 1y - 4z = 0 and 0x + 0y + 0z = -2.

Notice that 0x + 0y + 0z = -2 states 0 = -2 which is a FALSE statement. It indicates that this system has no solutions.

Graphically, any two or all three planes in 3D space could be parallel.

Example 8: Interpret a Solution Matrix

Discuss the following solution matrix of a system of three linear equations in three variables x, y, and z.

$$\begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 1 & -3 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$
 This translates to $1x + 1y - 3z = -1$ and $0x + 1y - z = 0$ and $0x + 0y + 0z = 0$.

Notice that 0x + 0y + 0z = 0 states 0 = 0 which is a TRUE statement. It indicates that this system has infinitely many solutions.

Graphically, the three planes lie on top of each other.