Examples
Logarithmic Equations in One Variable

Based on power point presentations by Pearson Education, Inc.

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Learning Objectives

1. Solve equations containing some logarithms.
2. Solve equations containing only logarithms.

Example 1: Solve Logarithmic Equations (1 of 2)

Solve $\log_2 (x - 4) = 3$. **Only find solutions that make the logarithm arguments positive!**

There already exists a single logarithm with coefficient 1. Therefore, we can change to exponential form as follows:

$x - 4 = 2^3$

$x - 4 = 8$

and $x = 12$ which is the proposed solution.

Example 1: Solve Logarithmic Equations (2 of 2)

Check $x = 12$ in the original equation $\log_2 (x - 4) = 3$ to ensure that the logarithm argument is positive.

$\log_2 (12 - 4) = 3$

and $\log_2 (8) = 3$

We see that the logarithm argument is positive. Therefore, we verified that $x = 12$ is an actual solution.

Example 2: Solve Logarithmic Equations (1 of 2)

Solve $4 \ln (3x) = 8$. **Only find solutions that make the logarithm arguments positive!**

Remember $\ln x$ equals $\log_e x$! Here we MUST first isolate the logarithmic expression to get a coefficient of 1. We do this by dividing both sides by 4.

We get $\ln (3x) = 2$.

Now we can change to exponential form as follows:

$\boldsymbol{e}^2 = 3x$

and $x = \frac{\boldsymbol{e}^2}{3}$

NOTE: In algebra, we usually do not change solutions to decimal form.

Example 2: Solve Logarithmic Equations (2 of 2)

Check $x = \frac{e^2}{3}$ in the original equation $4 \ln (3x) = 8$ to ensure that the logarithm argument is positive.

$$4 \ln \left(3 \left(\frac{e^2}{3} \right) \right) = 8$$

and $4 \ln (e^2) = 8$

We see that the logarithm argument is positive. Therefore, we verified that $x = \frac{e^2}{3}$ is an actual solution.

Example 3: Solve Logarithmic Equations (1 of 3)

Solve $\log x + \log(x - 3) = 1$. **Only find solutions that make the logarithm arguments positive!**

Remember that $\log x$ equals $\log_{10} x$. Here we MUST first achieve one single logarithm. We do this by using the *Product Rule* to get the following:

$\log(x(x - 3)) = 1$

Then we can continue to solve as follows:

$x(x - 3) = 10^1$ We changed to exponential form.

$x^2 - 3x = 10$ This is a quadratic equation.

and $x^2 - 3x - 10 = 0$

$(x - 5)(x + 2) = 0$ We factored this quadratic equation.

Example 3: Solve Logarithmic Equations (2 of 3)

We will now use the *Zero Product Principle* to solve for x.

$x - 5 = 0$ and $x + 2 = 0$

then $x = 5$ and $x = -2$, which are proposed solutions.

- Check $x = 5$ in the original equation $\log x + \log (x - 3) = 1$ to ensure that all logarithm arguments are positive.

  $\log (5) + \log (5 - 3) = 1$

  $\log (5) + \log (2) = 1$

  We see that all logarithm arguments are positive. Therefore, we verified that $x = 5$ is an actual solution.

Example 3: Solve Logarithmic Equations (3 of 3)

- Check $x = -2$ in the original equation $\log x + \log(x - 3) = 1$ to ensure that all logarithm arguments are positive.

$$\log(-2) + \log(-2 - 3) = 1$$

We see that both logarithm arguments are negative. Therefore, $x = -2$ is NOT a solution!

In summary, the only true solution is $x = 5$.

Example 4: Solve Logarithmic Equations

Solve $5\log(4x) = 10\log(x - 3)$. **Only find solutions that make the logarithm arguments positive!**

Here we must first achieve one single logarithm on the right and the left. Dividing both sides by 5 we get

$$\log(4x) = 2\log(x - 3)$$

Next, we will use the *Power Rule* to get the following:

$$\log(4x) = \log(x - 3)^2$$

Example 4: Solve Logarithmic Equations

Since both logarithms have base 10, we can state $4x = (x - 3)^2$.

Then, $4x = (x - 3)(x - 3)$ and $4x = x^2 - 6x + 9$.

We are dealing with a quadratic equation. We can solve it using the quadratic formula or possibly by factoring. In either case, we must have a 0 on one side of the equation.

Then $x^2 - 10x + 9 = 0$.

We find that we can factor the equation to get $(x - 1)(x - 9) = 0$.

Example 4: Solve Logarithmic Equations

We will now use the *Zero Product Principle* to solve for x.

$x - 1 = 0$ and $x - 9 = 0$

then $x = 1$ and $x = 9$, which are proposed solutions.

Now we need to check BOTH solutions in the original equation to ensure that the logarithm arguments are positive.

Example 4: Solve Logarithmic Equations (4 of 5)

- Check $x = 1$ in the original equation $5\log(4x) = 10\log(x - 3)$ to ensure that all logarithm arguments are positive.

  $5\log(4(1)) = 10\log(1 - 3)$

  $5\log(4) = 10\log(-2)$

  We see that one logarithm argument is negative. Therefore, $x = 1$ is NOT a solution.

Example 4: Solve Logarithmic Equations (5 of 5)

- Check $x = 9$ in the original equation $5\log(4x) = 10\log(x - 3)$ to ensure that all logarithm arguments are positive.

$$5\log(4(9)) = 10\log(9 - 3)$$

$$5\log(36) = 10\log(6)$$

We see that all logarithm arguments are positive. Therefore, we verified that $x = 9$ is an actual solution.

In summary, the only solution is $x = 9$.

Example 5: Solve a Logarithmic Equation (1 of 4)

Solve $\ln(x - 3) = \ln(7x - 23) - \ln(x + 1)$. **Only find solutions that make the logarithm arguments positive!**

Remember that $\ln x$ equals $\log_e x$. Here we must first write one single logarithmic expression on the right. We achieve this by using the *Quotient Rule*.

$$\ln(x-3) = \ln\left(\frac{7x-23}{x+1}\right)$$

Since both logarithms have base ***e***, we can state $x - 3 = \dfrac{7x-23}{x+1}$.

Example 5: Solve a Logarithmic Equation

We now multiply both sides by the denominator $(x + 1)$!

$$(x+1)(x-3)=\frac{7x-23}{x+1}\cdot(x+1)$$

$$(x-3)(x+1)=7x-23$$

Combining like terms we get $x^2 - 2x - 3 = 7x - 23$.

Since we are obviously dealing with a quadratic equation, we will prepare it for a solution by factoring or possibly by the quadratic formula.

That is, $x^2 - 9x + 20 = 0$.

We find that we can factor this to get $(x - 5)(x - 4) = 0$.

Example 5: Solve a Logarithmic Equation (3 of 4)

We will now use the *Zero Product Principle* to solve for x.

$x - 5 = 0$ and $x - 4 = 0$

then $x = 5$ and $x = 4$, which are proposed solutions.

- Check $x = 5$ in the original equation $\ln(x - 3) = \ln(7x - 23) - \ln(x + 1)$ to ensure that all logarithm arguments are positive.

$$\ln(5 - 3) = \ln(7 \cdot 5 - 23) - \ln(5 + 1)$$

and $\ln(2) = \ln(12) - \ln(6)$

We see that all logarithm arguments are positive. Therefore, we verified that $x = 5$ is an actual solution.

Example 5: Solve a Logarithmic Equation (4 of 4)

- Check $x = 4$ in the original equation $\ln(x - 3) = \ln(7x - 23) - \ln(x + 1)$ to ensure that all logarithm arguments are positive.

$$\ln(4 - 3) = \ln(7(4) - 23) - \ln(4 + 1)$$

and $\ln(1) = \ln(5) - \ln(5)$

We see that all logarithm arguments are positive. Therefore, we verified that $x = 4$ is an actual solution.

In summary, the equation has two solutions, namely $x = 4$ and $x = 5$.

Example 6: Solve a Logarithmic Equation (1 of 2)

Solve $\log\left(3 - \frac{1}{2}x\right) = \log(-x)$. **Only find solutions that make the logarithm arguments positive!**

Since both logarithms have base 10, we can state $3 - \frac{1}{2}x = -x$.

Then $3 = -x + \frac{1}{2}x$

and $3 = -\frac{1}{2}x$.

Finally, we will multiply both sides by $-\frac{2}{1}$ to get $x = -6$.

Example 6: Solve Logarithmic Equations (2 of 2)

Check $x = -6$ in the original equation $\log\left(3 - \frac{1}{2}x\right) = \log(-x)$ to ensure that the logarithm arguments are positive.

$$\log\left(3 - \frac{1}{2}(-6)\right) = \log(-(-6)).$$

and $\log(6) = \log(6)$

We see that the logarithm arguments are positive. Therefore, we verified that $x = -6$ is an actual solution.