Concepts
Linear Equations in One Variable

Based on power point presentations by Pearson Education, Inc.
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Learning Objectives

1. Memorize the *Basic Principles of Equations* consisting of four axioms.
2. Memorize what it means to solve an equation.
3. Solve linear equations containing integers and fractions.
4. Memorize and use the *Cross-Multiplication Principle*.

NOTE: This lesson contains some examples. You can find more examples in the “Examples” document also located in the appropriate MOM Learning Materials folder.

1. Basic Principles of Equations (1 of 3)

We were told previously that equations consist of mathematical expressions equal to other mathematical expressions. We now continue our study of equations, and the first thing we will learn is that any equation can be transformed into an equivalent equation by using the **Basic Principles of Equations**.

The **Basic Principles of Equations** consist of the following four axioms (statements that are self-evident and require no proof).

1. Addition Axiom – Given an equation, add the same quantity to **BOTH** sides of the equal sign.

For example, given $x - 9 = 3 + 2x$, add 9 to both sides of the equal sign. We write $x - 9 + 9 = 3 + 2x + 9$.

Basic Principles of Equations (2 of 3)

2. **Subtraction Axiom** – Given an equation, subtract the same quantity from **BOTH** sides of the equal sign.

   For example, given $4x + 5 = 1 + 3x$, subtract $3x$ from both sides of the equal sign. We write $4x + 5 - 3x = 1 + 3x - 3x$.

3. **Multiplication Axiom** – Given an equation, multiply **BOTH** sides of the equal sign by the same **nonzero** quantity.

   For example, given $3x = 9$, multiply both sides of the equal sign by $\frac{1}{3}$.

   We write $\frac{1}{3}(3x) = \frac{1}{3}(9)$.

Basic Principles of Equations (3 of 3)

4. **Division Axiom** – Given an equation, divide **BOTH** sides of the equal sign by the same **nonzero** quantity.

   For example, given $2x = 8$, divide both sides of the equal sign by 2.

   We write $\frac{2x}{2} = \frac{8}{2}$.

   NOTE: A fraction bar indicates division $\div$.

2. About Solving Equations

In mathematics, we encounter many different equations with one, two, or three variables, mostly using ***x***, ***y***, and ***z***. We usually define these equations in what is called the **general form** although in our work we often don't use it.

When solving equations containing variables, the goal is ALWAYS to "isolate" the variable. In mathematics, "to isolate" means that we want to end up with the variable by itself on one side of the equal sign having a coefficient of positive 1. The other side then gives us a number or numbers which is/are called "the solution(s)".

We accomplish "isolation" of the variable by using any combination of the *Addition*, *Subtraction*, *Multiplication*, and *Division Axioms*. We might have to use the axioms more than once.

3. Linear Equations in One Variable and their Solutions

One important set of equations are the **linear equation** in one variable, say ***x*** (Note: in mathematics we like to use x!) Its **general form** is as follows:

$\boldsymbol{Ax + B = 0}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are numbers with $\boldsymbol{A \neq 0}$

Please note that linear equations in one variable do not necessarily have to appear in *general form*. Mathematics just likes to define them that way!

Linear Equations in One Variable and their Solutions

Examples of linear equations in one variable:

a. $5x + 9 = 0$ (general form with $A = 5$ and $B = 9$)
   Please note that $5x$ means $5 \cdot x$.

b. $3x = 0$ (general form with $A = 3$ and $B = 0$)

c. $-4x - 6 = 0$

Note: The *General Form* requires a plus sign between the terms. Plus signs make no changes to the equations unlike minus signs.

Please note that this equation can be written as $-4x + (-6) = 0$. Now we see that $A = -4$ and $B = -6$. We simply eliminated the double signage to get the following:

$-4x - 6 = 0$

Linear Equations in One Variable and their Solutions (3 of 16)

We are now going to discuss strategies for two different types of linear equations in one variable. The first type only contains integers. The first type will also contain fractions.

A. Strategy for Solving Linear Equations Only Containing Integers

Step 1: If necessary, simplify the expressions on each side of the equal sign (remove grouping symbols, combine like terms, etc.).

Example 1:

Solve the linear equation $4(2x - 1) = 3(2x - 5) + 12$.

Note, this linear equation is NOT in *general form*!

$8x - 4 = 6x - 15 + 12$ Removed the parentheses by applying the Distributive Property!

$8x - 4 = 6x - 3$ Combined like terms on the right side of the equal sign!

Linear Equations in One Variable and their Solutions (4 of 16)

Step 2: If appropriate, always try to use the Addition and/or the Subtraction Axiom FIRST to begin "isolating" the variable. Specifically, move all variable terms to one side of the equal sign and all constants to the other side. Then combine any like terms.

Example 1 continued with $8x - 4 = 6x - 3$:

Let's get all variables to the left side of the equal sign. For this, we will subtract $\mathbf{6x}$ from both sides (Subtraction Axiom). Then we will combine any like terms.

$8x - 4 - 6x = 6x - 3 - 6x$

and $2x - 4 = -3$ Combined like terms on the right and left side of the equal sign!

Linear Equations in One Variable and their Solutions

Example 1 continued:

Given $2x - 4 = -3$, let's now get all constants to the right side of the equal sign. For this, we will add ***4*** to both sides (Addition Axiom). Then we will combine any like terms.

$2x - 4 + 4 = -3 + 4$

and $2x = 1$ Combined like terms on the right and left side of the equal sign!

Linear Equations in One Variable and their Solutions

Step 3: If appropriate, further "isolate" the variable by using the Multiplication and/or the Division Axiom. This means the coefficient of the variable MUST be + 1 (never –1). *This is will be the proposed solution of the equation.*

Example 1 continued with $2x = 1$:

Given $2x = 1$, we will "isolate" the variable by dividing both sides of the equation by ***2*** (which is the coefficient of the variable). That is, we are using the Division Axiom!

$$\frac{\overset{1}{\cancel{2}}x}{\underset{1}{\cancel{2}}} = \frac{1}{2}$$

$$x = \frac{1}{2}$$ *This is the proposed solution.*

NOTE: Algebra prefers fractions to decimal. Therefore, avoid stating $x = 0.5$.

Linear Equations in One Variable and their Solutions (7 of 16)

Step 4: Prove that the proposed solution is an actual solution. We do this by replacing the variable in the original equation with the proposed solution. If the result is a true statement, then the proposed solution is an actual solution.

Example 1 continued:

Given $x = \frac{1}{2}$, we replace x in the original equation $4(2x - 1) = 3(2x - 5) + 12$ with $\frac{1}{2}$.

$$
\begin{aligned}
4\left(2\left(\tfrac{1}{2}\right) - 1\right) &= 3\left(2\left(\tfrac{1}{2}\right) - 5\right) + 12 \\
4(1 - 1) &= 3(1 - 5) + 12 \\
4(0) &= 3(-4) + 12 \\
4(0) &= -12 + 12 \\
0 &= 0
\end{aligned}
$$

The last equation is a true statement, therefore the proposed solution $x = \frac{1}{2}$ is an actual solution.

Linear Equations in One Variable and their Solutions

B. Strategy for Solving Linear Equations Containing Fractions

When we encounter fractions in a linear equation, we usually want to quickly ”eliminate” them to make our job of solving the equation easier. Following is a strategy on how to do this.

Step 1: Find a number evenly divisible (no remainder) by all denominators of the fractions in the equation. Although it makes no difference in the end, we usually prefer to find the smallest such number because the subsequent calculations may not be as cumbersome.

Linear Equations in One Variable and their Solutions (9 of 16)

Example 2:

Solve the linear equation $\frac{x-3}{4} = \frac{5}{14} - \frac{x+5}{7}$.

Note, this linear equation is NOT in *general form*!

Let’s find a number evenly divisible by all denominators. Please understand that we can always find such a number by calculating the product of all denominators. In our case, that would be 4(14)(7) = 392.

However, we might notice that **28** is also a number evenly divisible by 4, 14, and 7. It certainly is a lot smaller than 392 and therefore easier to work with. Actually, it is the smallest number evenly divisible by all denominators.

Linear Equations in One Variable and their Solutions

Step 2: Multiply both sides of the equal sign by the number found in Step 1 and solve for the variable. *This is the proposed solution to the equation.*

Example 2 continued:

Let’s use 28, the smallest number divisible by all denominators and multiply both sides of the equal sign by it.

$$28\left(\frac{x-3}{4}\right) = 28\left(\frac{5}{14} - \frac{x+5}{7}\right)$$

Using the *Distributive Property*, we get the following:

$$28\left(\frac{x-3}{4}\right) = 28\left(\frac{5}{14}\right) + 28\left(-\frac{x+5}{7}\right)$$

Linear Equations in One Variable and their Solutions (11 of 16)

Example 2 continued:

Before we continue, let's swap the negative sign of the second term on the right side of the equal sign with the positive sign of 28! This is acceptable and allows for easier calculations.

$$28\left(\frac{x-3}{4}\right) = 28\left(\frac{5}{14}\right) + 28\left(-\frac{x+5}{7}\right)$$

$$28\left(\frac{x-3}{4}\right) = 28\left(\frac{5}{14}\right) - 28\left(+\frac{x+5}{7}\right)$$

NOTE: The leading plus sign in the parentheses of the second term on the right side of the equal sign does not have to be written.

We end up with the following:

$$\overset{7}{\cancel{28}}\left(\frac{x-3}{\cancel{4}\,1}\right) = \overset{2}{\cancel{28}}\left(\frac{5}{\cancel{14}\,1}\right) - \overset{4}{\cancel{28}}\left(\frac{x+5}{\cancel{7}\,1}\right)$$

We simplified the fractions, that is, we wrote them in lowest terms.

Linear Equations in One Variable and their Solutions

Example 2 continued with $\overset{7}{\cancel{28}}\left(\frac{x-3}{\underset{1}{\cancel{4}}}\right) = \overset{2}{\cancel{28}}\left(\frac{5}{\underset{1}{\cancel{14}}}\right) - \overset{4}{\cancel{28}}\left(\frac{x+5}{\underset{1}{\cancel{7}}}\right)$:

After reducing the fractions to lowest terms, we can write
$7(x-3) = 2(5) - 4(x+5)$.

> Please not that the numerators $(x-3)$ and $(x+5)$ must be in parentheses!

Using the *Distributive Property*, we find $7x - 21 = 10 - 4x - 20$

and combining like terms, we get $7x - 21 = -4x - 10$.

Linear Equations in One Variable and their Solutions

Step 3: If appropriate, always try to use the Addition and/or the Subtraction Axiom FIRST to begin "isolating" the variable. Specifically, move all variable terms to one side of the equal sign and all constants to the other side. Then combine any like terms.

Example 2 continued with $7x - 21 = -4x - 10$: :

Using the Addition Axiom and combining like terms we end up with with the following:

$11x = 11$

Linear Equations in One Variable and their Solutions

Step 4: If appropriate, further “isolate” the variable by using the Multiplication and/or the Division Axiom. This means the coefficient of the variable MUST be + 1 (never –1). *This is will be the proposed solution of the equation.*

Example 1 continued with $11x = 11$:

Given $11x = 11$, we will “isolate” the variable by dividing both sides of the equation by ***11*** (which is the coefficient of the variable). That is, we are using the Division Axiom! We will then reduce the fractions.

$$\frac{{}^{1}\,\cancel{11}x}{{}_{1}\,\cancel{11}} = \frac{\cancel{11}^{\,1}}{\cancel{11}_{\,1}}$$

and $x = 1$. This is the proposed solution.

Linear Equations in One Variable and their Solutions

Step 5: Replace the variable in the original equation with the proposed solution. If the result is a true statement, then the proposed solution is an actual solution.

Example 2 continued:

Let's replace *x* in the original equation $\frac{x-3}{4}=\frac{5}{14}-\frac{x+5}{7}$ with 1.

$$\frac{1-3}{4}=\frac{5}{14}-\frac{1+5}{7}$$

$$\frac{-2}{4}=\frac{5}{14}-\frac{6}{7}$$

NOTE: When we add or subtract fractions with "unlike" denominators, we must change all denominators to the same number. We will always use a number divisible by all denominators without a remainder.

Linear Equations in One Variable and their Solutions

Example 2 continued:

So, let’s use 28 again and multiply both sides of the equal sign by it.

$$28\left(\frac{-2}{4}\right) = 28\left(\frac{5}{14}\right) + 28\left(-\frac{6}{7}\right)$$

We will simply the fractions as follows

$$\overset{7}{\cancel{28}}\left(\frac{-2}{\cancel{4}_{1}}\right) = \overset{2}{\cancel{28}}\left(\frac{5}{\cancel{14}_{1}}\right) + \overset{4}{\cancel{28}}\left(-\frac{6}{\cancel{7}_{1}}\right)$$

and get 7 (– 2) = 2 (5) + 4 (– 6)

Finally, we end up with – 14 = 10 – 24 , and we see that – 14 = – 14.

The last equation is a true statement, therefore the proposed solution x = 1 is an actual solution.

4. Cross Multiplication (1 of 3)

A special method called **Cross-Multiplication** can often be used when we encounter equations with fractions IFF there is exactly one fraction is equal to another fraction. These type of equations are also called *proportions*!

For example, $\frac{F}{3} = \frac{9}{5}$ is a linear equation and its variable is ***F***.

We will now make a “cross” × over the equal sign! That is, $\frac{F}{3} \times \frac{9}{5}$.

Please note that *Cross-Multiplication* is a process of multiplying the numerator of one fraction with the denominator of the other fraction on the other side of an equal sign.

According to the *Cross-Multiplication* process, we can now state ***5F*** = ***3(9)*** or ***5F*** = ***9(3)***.

Notice that we wrote ***5F*** and not ***F5*** or ***F(5)***! We always place the coefficient in front of the variable!

Cross-Multiplication (2 of 3)

Let's finish solving $\frac{F}{3} = \frac{9}{5}$.

Simplifying $\boldsymbol{5F = 3(9)}$, we get $\boldsymbol{5F = 27}$. We will now proceed with the solution strategy for linear equations involving integers to solve for $\boldsymbol{F}$.

We divide both sides of the equal sign by $\boldsymbol{5}$ and reduce the fraction on the left to get the following:

$$\frac{\overset{1}{\cancel{5}}F}{\underset{1}{\cancel{5}}} = \frac{27}{5}$$

and $F = \frac{27}{5}$ This is a proposed solution!

NOTE: While $\frac{27}{5}$ can be written as $5\,\frac{2}{5}$ or 5.4, in algebra, we usually do not express solutions as mixed numbers or decimals. So, we will keep $F = \frac{27}{5}$.

Cross-Multiplication (3 of 3)

If we wish, we could carry out a check to prove to ourselves that $\frac{27}{5}$ is indeed the actual solution.

But the solution is a bit “nasty” so we will just convince ourselves that the proposed solution is the true solution. That’s something we can do when we solve linear equations.