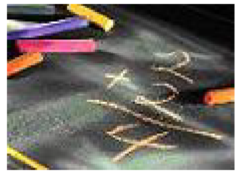

DETAILED SOLUTIONS AND CONCEPTS - THE SIX TRIGONOMETRIC RATIOS

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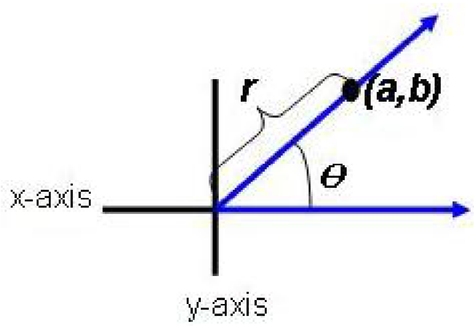
Please Send Questions and Comments to ingrid.stewart@csn.edu. Thank you!

PLEASE NOTE THAT YOU CANNOT ALWAYS USE A CALCULATOR ON THE ACCUPLACER - COLLEGE-LEVEL MATHEMATICS TEST! YOU MUST BE ABLE TO DO SOME PROBLEMS WITHOUT A CALCULATOR!

Let's construct an angle θ in **standard position** in the *Cartesian Coordinate System*. Then we'll place a point ***(a, b)*** on the terminal side of the angle a distance $r = \sqrt{a^2 + b^2}$ away from its vertex.



For simplicity's sake, the picture shows an acute angle . Please be aware that the following definitions of the six trigonometric ratios hold for angles of any magnitude!!

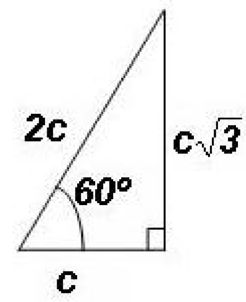
The numeric values of the six trigonometric ratios are then defined as follows. **Please note that they are NOT defined by the magnitude of the angle!**

Sine Ratio: $\sin\theta = \dfrac{b}{r}$ Pronounce ***sin*** θ as "sine theta".	Cosine Ratio: $\cos\theta = \dfrac{a}{r}$ Pronounce ***cos*** θ as "cosine theta".
Tangent Ratio: $\tan\theta = \dfrac{b}{a} \quad a \neq 0$ Pronounce ***tan*** θ as "tangent theta".	Cosecant Ratio: $\csc\theta = \dfrac{r}{b}$ Pronounce ***csc*** θ as "cosecant theta" (koseekent theta).
Secant Ratio: $\sec\theta = \dfrac{r}{a}$ Pronounce ***sec*** θ as "secant theta" (seekent theta) .	Cotangent Ratio: $\cot\theta = \dfrac{a}{b} \quad b \neq 0$ Pronounce ***cot*** θ as "cotangent theta".

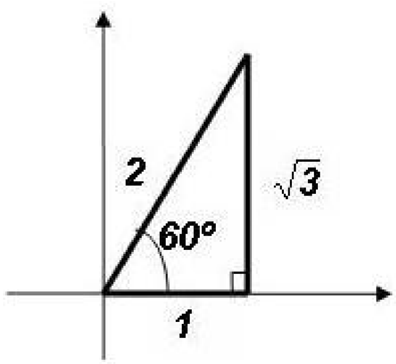
Note that secant, cosecant, and cotangent are reciprocals of the cosine, sine, and tangent, respectively!!!

To illustrate the definitions of the trigonometric ratios, let's use the *60º* angle.

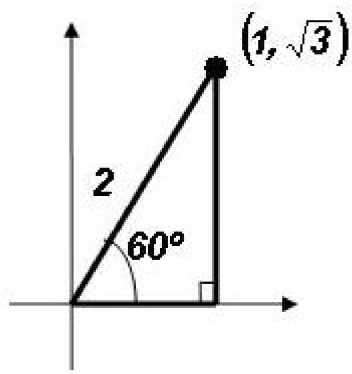
We are going to use the ***30º- 60º- 90º*** triangle. We know from our high school Geometry class that its sides are in the proportions $c, c\sqrt{3}, 2c$, where ***2c*** is the length of the *hypotenuse* of the triangle. Recall that the *hypotenuse* is the side opposite the right angle!



Let's place the ***30º- 60º- 90º*** triangle into the first quadrant of a coordinate system, so that the ***60º*** angle has its vertex at the origin. For simplicity's sake, we will let ***c = 1***, although, ***c*** could take on any value!!!



and



Using the definition of the trigonometric ratios from above, you can now convince yourself that the EXACT numeric values of the six trigonometric ratios of the angle ***60º*** are

$\sin 60^\circ = \frac{\sqrt{3}}{2}$ $\cos 60^\circ = \frac{1}{2}$ $\tan 60^\circ = \sqrt{3}$

$\csc 60^\circ = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$ $\sec 60^\circ = \frac{2}{1} = 2$ $\cot 60^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

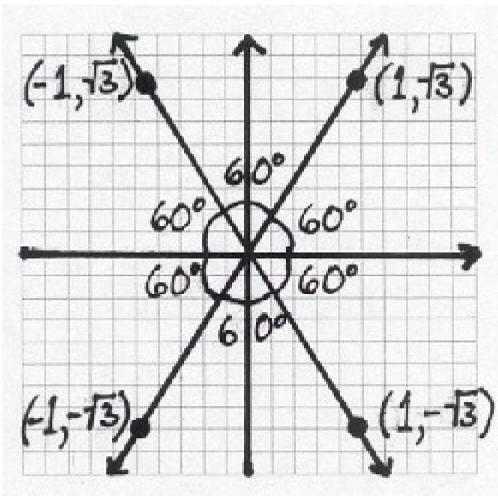
Incidentally, you are always supposed to rationalize the denominator! The correct answer for the cosecant and cotangent is the last number! It is NOT standard form to keep a radical in the denominator!!!

- Please note that in the example, all numeric values are positive since a point ***(a, b)*** in the first quadrant has a positive ***a*** and ***b*** and ***r***, a distance, is always positive.
- On the other hand, if we chose a point ***(a, b)*** in the second quadrant, it has a negative ***a*** and a positive ***b***. Therefore, the sine and cosecant ratios have positive numeric values for second-quadrant angles, while the other four ratios take on negative values.
- Likewise, if we chose a point ***(a, b)*** in the third quadrant, it has both a negative ***a*** and ***b***.

Therefore, the tangent and cotangent ratios have positive numeric values for third-quadrant angles, while the other four ratios take on negative values.

- Finally, if we chose a point ***(a, b)*** in the fourth quadrant, it has a positive ***a*** and a negative ***b***. Therefore, the cosine and secant ratios have positive numeric values for fourth-quadrant angles, while the other four ratios take on negative values.

Now let's copy the ***30º- 60º- 90º*** triangle into every quadrant as follows:



Using the hypotenuse as the terminal side of an angle, you will find that all angles drawn in that manner (positive and negative have a reference angle of ***60º***.

Let's investigate numeric values of some angles using the cosine ratio!

cos 60º = $\frac{1}{2}$ ***cos 120º =*** $-\frac{1}{2}$

cos 240º = $-\frac{1}{2}$ ***cos 300º =*** $\frac{1}{2}$

cos (-60º) = $\frac{1}{2}$ ***cos (-120º) =*** $-\frac{1}{2}$

cos (-240º) = $-\frac{1}{2}$ ***cos (-300º) =*** $\frac{1}{2}$

cos 600º = $-\frac{1}{2}$ ***cos (-660º) =*** $\frac{1}{2}$

- Observe that just because an angle is negative does not mean that the numeric value of the trigonometric ratio must be negative.
- Likewise, a positive angle does not necessarily have a positive numeric value.
- However, the trigonometric ratio of the angles and the trigonometric ratio of their reference angle (60º) have the same *absolute* numeric value.

We will now use our work with the *60º* angle to derive the following important concepts that can be generalized to any angle!

1. Trigonometric Identities

In trigonometry, a great deal of time is spent studying relationships between trigonometric ratios. We call these relationships "identities." The following identities will be used many times in trigonometry and later in calculus. Learn them well !!!

Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Please note that the *Reciprocal* and *Quotient Identities* will hold for any trigonometric function raised to a power. For example,

$$\sin^2 x = \frac{1}{\csc^2 x} = \left(\frac{1}{\csc x}\right)^2$$

or

$$\tan^3 x = \frac{\sin^3 x}{\cos^3 x} = \left(\frac{\sin x}{\cos x}\right)^3$$

NOTE: $\sin^2 x = (\sin x)^2 = \sin x \cdot \sin x$**. This applies to the other trigonometric functions also.**

Observe that $\sin^2 x \neq \sin x^2$ **!!!!!!! However,** $\sin x^2 = \sin(x^2)$**.**

Identities for Negatives

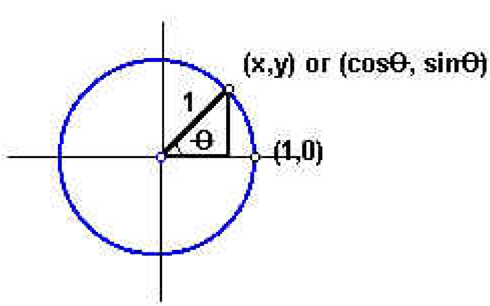
$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta$$

$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$

$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

Pythagorean Identities

They are derived from the following unit circle



Using the definitions of sine and cosine above we find that in the unit circle $x = \cos\theta$ and $y = \sin\theta$.

Therefore, we can rewrite the equation of the unit circle $x^2 + y^2 = 1$ and derive what is called a **Pythagorean Identity**. That is,

1. $\sin^2\theta + \cos^2\theta = 1$

   It can also be expressed as $\cos^2\theta = 1 - \sin^2\theta$ or $\sin^2\theta = 1 - \cos^2\theta$

2. $1 + \tan^2\theta = \sec^2\theta$

   This is another form of the *Pythagorean Identity* derived from dividing every term of $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$.

   It can also be expressed as $\tan^2\theta = \sec^2\theta - 1$ or $1 = \sec^2\theta - \tan^2\theta$

3. $1 + \cot^2\theta = \csc^2\theta$

   This is yet another form of the *Pythagorean Identity* derived from dividing every term in $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$.

   It can also be expressed as $\cot^2\theta = \csc^2\theta - 1$ or $1 = \csc^2\theta - \cot^2\theta$

2. Unique "sign" Pattern of the Numeric Values

This "sign" pattern can be summarized in the following table. A handy memorization aid to remember this by is: **All Students Take Calculus**

Terminal side of the angle is in Quadrant I	**All** trigonometric ratios have positive numeric values.
Terminal side of the angle is in Quadrant II	**S**ine and cosecant ratios have positive numeric values, all others have negative values.
Terminal side of the angle is in Quadrant III	**T**angent and cotangent ratios have positive numeric values, all others have negative values.
Terminal side of the angle is in Quadrant IV	**C**osine and secant ratios have positive numeric values, all others have negative values.

3. Relationship between the Trigonometric Ratios of an Angle and the Trigonometric Ratio of its Reference Angle

The trigonometric ratios of an angle and the trigonometric ratio of its reference angle (if it exists) have the same absolute numeric value.

Problem 1:

The point ***(-3, 4)*** lies on the terminal side of an angle θ , which is in standard position. Determine the EXACT numeric values of the six trigonometric ratios of the angle θ.

Using the definitions for the six trigonometric ratios, we let ***a = -3*** and ***b = 4***. Then $r = \sqrt{(-3)^2 + 4^2} = 5$.

Therefore,

$$\cos\theta = \frac{a}{r} = \frac{-3}{5} = -\frac{3}{5}$$

Using the *Reciprocal Identity,* we find that $\sec\theta = -\frac{5}{3}$

$$\sin\theta = \frac{b}{r} = \frac{4}{5}$$

Using the *Reciprocal Identity,* we find that $\csc\theta = \frac{5}{4}$

$$\tan\theta = \frac{b}{a} = \frac{4}{-3} = -\frac{4}{3}$$

Using the *Reciprocal Identity,* we find that $\cot\theta = -\frac{3}{4}$

Problem 2:

Indicate in which two quadrants the terminal side of the following angles θ must lie. Assume that θ is not a *Quadrantal Angle*.

For this we will use "All Students Take Calculus."

a. $\cos\theta = -\frac{\sqrt{5}}{6}$ Quadrant II and III	b. $\tan\theta = \frac{1}{4}$ Quadrant I and III	c. $\cot\theta = -\sqrt{7}$ Quadrant II and IV	d. $\sin\theta = -\frac{2}{3}$ Quadrant III and IV
e. $\cos\theta = \frac{2}{5}$ Quadrant I and IV	f. $\tan\theta = -3$ Quadrant II and IV	g. $\cot\theta = \frac{\sqrt{11}}{9}$ Quadrant I and III	h. $\sin\theta = \frac{\sqrt{2}}{2}$ Quadrant I and II

Problem 3:

True or False? If $\cos 45^\circ = \frac{\sqrt{2}}{2}$, then $\cos 135^\circ = -\frac{\sqrt{2}}{2}$.

True, since the angle ***135º*** has a reference angle of ***45º***, we know that both ratios must have the same absolute numeric value. However, using **All Students Take Calculus** we know the numeric value of ***cos 135º*** must be negative.

While it is beneficial to know how the six trigonometric ratios are defined, we will mostly use our calculator to actually find the numeric values of the trigonometric ratios. Incidentally, the calculator uses a program that computes the numeric values using concepts from advanced mathematics!

Your calculator has a ***sin***, ***cos***, and ***tan*** key. It does NOT have a key for ***csc***, ***sec***, and ***cot*** !!! You MUST find the numeric values of secant, cosecant, and cotangent on your calculator by using the *Reciprocal* and *Quotient Identities*.

NOTE: The $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ feature on your calculator does NOT calculate cosecant, secant and cotangent !!! These keys calculate the inverse sine, inverse cosine, and inverse tangent, which will be discussed later.

Degree and Radian Measure on your Calculator:

Your calculator must be told whether it is supposed to find numeric values of trigonometric ratios with angles in degree or radians. It is very important that you change the calculator mode to either radians or degrees. Not doing so, will give you false information. This is especially disastrous on a test!

- Scientific calculator:

  Your scientific calculators most likely has a ***DRG*** key. Press this key and observe a ***D***, ***R***, or ***G*** in the display window. ***D*** means that your calculator is in degree mode. ***R*** means that your calculator is in radian mode. ***G*** means that your calculator is in gradient mode (we don't need this one!).

- Graphing calculator:

  Your graphing calculator most likely has a ***MODE*** feature in which you can change from degrees to radians and vice versa. Unfortunately, some graphing calculators do not indicate in their display window what mode they are in. Therefore, before you carry out any calculations, make sure to check the mode!!!

It is often important in the real world to work with EXACT numbers. In the table below find the EXACT numeric values of the special angle measures ***0º***, ***90º***, ***180º***, ***270º***, ***360º***, ***30º***, ***45º*** and ***60º***. **You must memorize these values!**

x	$\sin x$	$\cos x$	$\tan x$	$\csc x$	$\sec x$	$\cot x$
$0° \equiv 0$	0	1	0	undefined	1	undefined
$90° \equiv \frac{\pi}{2}$	1	0	undefined	1	undefined	0
$180° \equiv \pi$	0	-1	0	undefined	-1	undefined
$270° \equiv \frac{3\pi}{2}$	-1	0	undefined	-1	undefined	0
$360° \equiv 2\pi$	0	1	0	undefined	1	undefined
$30° \equiv \frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$45° \equiv \frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$60° \equiv \frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

Please observe the following decimal approximations for the EXACT numeric values of trigonometric ratios of some special angles:

$\frac{\sqrt{3}}{2} \approx 0.866$ $\frac{\sqrt{2}}{2} \approx 0.707$ $\frac{\sqrt{3}}{3} \approx 0.577$

$\sqrt{3} \approx 1.732$ $\frac{2\sqrt{3}}{3} \approx 1.155$ $\sqrt{2} \approx 1.414$

If you memorize these approximations, you do NOT have to worry about forgetting the EXACT values! That is, use your calculator to find the decimal value of the trigonometric ratio of the "special" angle and then convert this value into EXACT values.

Given the values in the table above, we can also find EXACT numeric values of trigonometric ratios of angles

- whose reference angle measure is the special angle measures 30^o, 45^o, or 60^o
- whose measures are integer multiples of quadrantal angles

Problem 4:

Use a calculator to find the numeric value of $\sin 34^o$ rounded to four decimal places.

$\sin 34^o \approx 0.5592$

Problem 5:

Use a calculator to find the numeric value of $\cos 55^o$ rounded to four decimal places.

$\cos 55^o \approx 0.5736$

Problem 6:

Use a calculator to find the numeric value of $\cos\left(\frac{\pi}{8}\right)$ rounded to three decimal places.

cos	(	π	÷	8	)	Enter

NOTE: Always use the π symbol on your calculator and not 3.14.

$\cos\left(\frac{\pi}{8}\right) \approx 0.924$

Question:

Would you get the same result if you were to do the following?

cos	π	÷	8	Enter

instead of

cos	(	π	÷	8	)	Enter

No, you found $\frac{\cos\pi}{8} = -\frac{1}{8}$ in the first calculation and $\cos\left(\frac{\pi}{8}\right)$ in the second one (see Example 3).

Problem 7:

Use a calculator to find the numeric value of $\sec 13^o$ rounded to three decimal places.

You MUST use $\sec\theta = \frac{1}{\cos\theta}$. You also have to make sure that your calculator is in degree mode.

1	÷	cos	(	13	)	Enter

$\sec 13^o \approx 1.026$

Problem 8:

Use a calculator to find the numeric value of ***csc 39º*** rounded to three decimal places.

You MUST use $\csc\theta = \dfrac{1}{\sin\theta}$. You also have to make sure that your calculator is in degree mode.

1	÷	*sin*	*(*	*39*	*)*	*Enter*

csc 39º ≈ 1.589

Problem 9:

Find the numeric value of ***tan 90º***.

tan 90º is undefined

Please note the following:

From the *Quotient Identity* we know that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

In our case,

$$\tan 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ} = \frac{1}{0} = \textit{undefined}$$

Problem 10:

Find the numeric value of ***cot 90º***.

You MUST use $\cot\theta = \dfrac{1}{\tan\theta}$ or $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

Please note, that in the case of cotangents and quadrantal angles it is BETTER to use the *Quotient Identity* since some calculators are not yet equipped to handle the *Reciprocal Identity*.

Input cos(90)/sin(90).

cot 90º = 0

Incidentally, had you used the *Reciprocal Identity* $\cot\theta = \frac{1}{\tan\theta}$ many calculators would have NOT been able to deal with $\frac{1}{\tan 90^o} = \frac{1}{\text{undefined}}$ (see example above) and would have given an Error Message!!!

Problem 11:

Use a calculator to find the numeric value of ***cot 1*** rounded to four decimal places.

Make sure that your calculator is in radian mode.

$\cot 1 \approx 0.6421$

Problem 12:

Find the EXACT numeric value of $\sin 315^o$.

Using the calculator we find that $\sin 315^o \approx -0.707$.

315^o has a reference angle of 45^o and we have memorized that $\sin 45^o = \frac{\sqrt{2}}{2} \approx 0.707$.

Therefore, $\sin 315^o = -\frac{\sqrt{2}}{2}$ exactly.

Problem 13:

Find the EXACT numeric value of $\csc 225^o$.

Using the calculator we find that $\csc 225^o \approx -1.414$.

225^o has a reference angle of 45^o and we have memorized that $\csc 45^o = \sqrt{2} \approx 1.414$.

Therefore, $\csc 225^o = -\sqrt{2}$ exactly.

Problem 14:

Find the EXACT numeric value of $\sec 330^o$.

Using the calculator we find that $\sec 330^o \approx 1.155$.

330^o has a reference angle of 30^o and we have memorized that $\sec 30^o = \frac{2\sqrt{3}}{3} \approx 1.155$.

Therefore, $\sec 330^o = \frac{2\sqrt{3}}{3}$ exactly.

Problem 15:

Use a calculator to find the numeric value of $\csc\left(-\frac{11\pi}{13}\right)$. Round to 3 decimal places.

$$\csc\left(-\frac{11\pi}{13}\right) = \frac{1}{\sin\left(-\frac{11\pi}{13}\right)} \approx -2.152$$

Problem 16:

Find the EXACT numeric value of $\cos 150^o$.

Using the calculator we find that $\cos 150^o \approx -0.866$.

We know that the trigonometric ratio of an angle and the trigonometric ratio of its reference angle have the same absolute numeric value.

150^o has a reference angle of 30^o and we have memorized that

$\cos 30^o = \frac{\sqrt{3}}{2} \approx 0.866$.

Therefore, $\cos 150^o = -\frac{\sqrt{3}}{2}$ exactly.

Problem 17:

Find the EXACT numeric value of $\tan 210^o$.

Using the calculator we find that $\tan 210^o \approx 0.577$.

210^o has a reference angle of 30^o and we have memorized that

$\tan 30^o = \frac{\sqrt{3}}{3} \approx 0.577$.

Therefore, $\tan 210^o = \frac{\sqrt{3}}{3}$ exactly.

Problem 18:

Find the EXACT numeric value of $\sin(-135^o)$.

Using the calculator we find that $\sin(-135^o) \approx -0.707$.

-135^o has a reference angle of 45^o and we have memorized that

sin 45º = $\frac{\sqrt{2}}{2}$ ***≈ 0.707***.

Therefore, ***sin (-135º) =*** $-\frac{\sqrt{2}}{2}$ exactly.

Problem 19:

Find the EXACT numeric value of ***tan 300º***.

Using the calculator we find that ***tan 300º ≈ -1.732***.

300º has a reference angle of ***60º*** and we have memorized that ***tan 60º*** = $\sqrt{3}$ ***≈ 1.732***.

Therefore, ***tan 300º =*** $-\sqrt{3}$ exactly.

Problem 20:

Find the EXACT numeric value of ***sin (11π/6)***.

Using the calculator in radian mode we find that ***sin (11π/6) =*** $-\frac{1}{2}$.

The numeric value is already exact!

Problem 21:

Find the EXACT numeric value of ***tan 225º***.

Using the calculator we find that ***tan 225º = 1***.

The numeric value is already exact!

Problem 22:

Find the EXACT numeric value of ***sin 660º***.

Using the calculator we find that ***sin 660º ≈ - 0.866***.

660º has a reference angle of ***60º*** and we have memorized that ***sin 60º*** = $\frac{\sqrt{3}}{2}$ ***≈ 0.866***.

Therefore, ***sin 660º =*** $-\frac{\sqrt{3}}{2}$ exactly.

Problem 23:

Find the EXACT numeric value of ***cos (-15π/4)***.

Using the calculator in radian mode we find that ***cos (-15π/4)*** $\approx$ ***0.707***.

-15π/4 has a reference angle of ***π/4*** and we have memorized that ***cos (π/4) =*** $\frac{\sqrt{2}}{2}$ $\approx$ ***0.707***.

Therefore, ***cos (-15π/4) =*** $\frac{\sqrt{2}}{2}$ exactly.

Problem 24:

Find the EXACT numeric value of ***cot 90º***.

Using the calculator we find that $\cot 90° = \frac{\cos 90°}{\sin 90°} = \frac{0}{1} = 0$, which is already exact.

NOTE: If we use the *Reciprocal Identity*, we get

$$\cot 90° = \frac{1}{\tan 90°} = \frac{1}{\text{undefined}}$$

Many "unsophisticated" calculators would give you an "Error Message" instead of ***0***. Therefore, it is better to use the *Quotient Identity* instead of the *Reciprocal Identity* when finding the numeric values of the cotangent.

Problem 25:

Find the EXACT numeric value of ***sin 630º***.

Using the calculator we find that ***sin 630º = -1***, which is already exact.

Problem 26:

Find the EXACT numeric value of ***sin(-90º)***.

Using the calculator we find that ***sin(-90º) = -1***, which is already exact.

Problem 27:

Find the EXACT numeric value of ***$\cos(-540^o)$***.

Using the calculator we find that ***$\cos 540^o = -1$***, which is already exact.