DETAILED SOLUTIONS AND CONCEPTS - GRAPHS OF SINE AND COSINE FUNCTIONS

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Please Send Questions and Comments to ingrid.stewart@csn.edu. Thank you!

PLEASE NOTE THAT YOU CANNOT ALWAYS USE A CALCULATOR ON THE ACCUPLACER - COLLEGE-LEVEL MATHEMATICS TEST! YOU MUST BE ABLE TO DO SOME PROBLEMS WITHOUT A CALCULATOR!

Definition of the Sine and Cosine Functions

$f(x) = \sin x$ and $g(x) = \cos x$

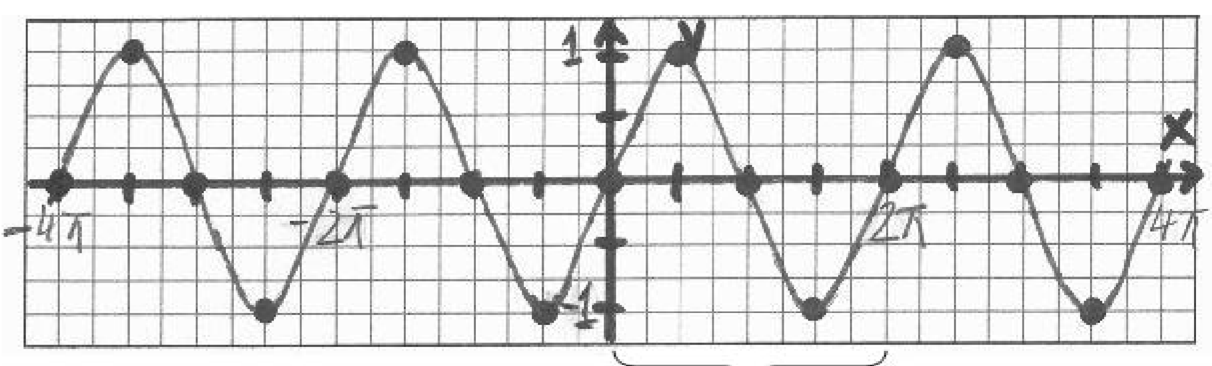
The *Sine* and *Cosine Functions* are called "periodic" with *period* $2\pi \approx 6.28$.

The domain of both functions (the x-values) contains all real numbers, which can be expressed as $(-\infty, \infty)$ in *Interval Notation*.

Please note that the values in the domain of ALL trigonometric functions are always in radian measure. NO degree measure can be used in the domain of a function, that is, for the x-values.

Characteristics of Graphs of the Sine Function

You can either take my word that the graph of the *Sine Function* appears as below or you can convince yourself by utilizing point-by-point plotting!



Representative Picture

Please observe the following:

Given $f(x) = \sin x$, we can create the following table of numbers:

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin x$	0	1	0	-1	0

. . . remember from your prerequisite algebra course . . . $f(0) = \sin(0) = 0$ **. . .** $f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$ **. . . use your calculator, if necessary!**

Now let's look at the graph above and make some observations:

a. The ***representative picture*** of the graph of $f(x) = \sin x$ lies on the interval ***[0, 2π]***. It repeats itself "forever" along the positive and negative x-axis creating the sine wave. **Please note that its peaks and valleys are U-shaped!**

b. Obviously then, it takes a distance of $2\pi \approx 6.28$ along the x-axis for the *representative picture* to be created. This distance is called the ***period***, which is considered one ***cycle*** of the sine function.

c. The *representative picture* of the sine function starts at ***(0,0)***.

d. The *representative picture* appears to be divided into **four equal intervals**.

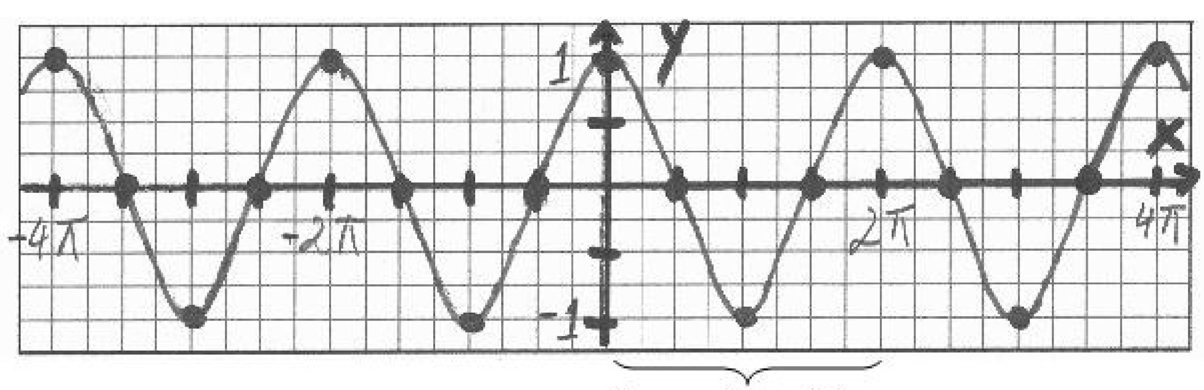
- At the end of the first interval the graph has a peak. Its coordinates are ***(π/2, 1)***.
- At the end of the second interval the graph has an x-intercept. Its coordinates are ***(π, 0)***.
- At the end of the third interval, the graph has a valley. Its coordinates are ***(3π/2, -1)***.
- At the end of the fourth interval the graph has an x-intercept. Its coordinates are ***(2π, 0)***.

Characteristics of Graphs of the Cosine Function

Again, you can either take my word that the graph of the *Cosine Function* appears as below or you can convince yourself by utilizing point-by-point plotting!

For example,

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos x$	1	0	-1	0	1



Representative Picture

Let's look at the graph above and make some observations:

a. The ***representative picture*** of the graph of $g(x) = \cos x$ also lies on the interval $[0, 2\pi]$. It also repeats itself "forever" along the positive and negative x-axis creating the cosine wave. **Please note that its peaks and valleys are U-shaped!**

b. Again obviously, it takes a distance of $2\pi \approx 6.28$ along the x-axis for the *representative picture* to be created. This distance is again called the ***period***, which is considered one ***cycle*** of the cosine function.

c. The *representative picture* of the cosine function starts at $(0,1)$.

d. The *representative picture* appears to be divided into **four equal intervals**.

- At the end of the first interval the graph has an x-intercept. Its coordinates are $(\pi/2, 0)$.
- At the end of the second interval the graph has a valley. Its coordinates are $(\pi, -1)$.
- At the end of the third interval, the graph has an x-intercept. Its coordinates are $(3\pi/2, 0)$.
- At the end of the fourth interval the graph has a peak. Its coordinates are $(2\pi, 1)$.

Strategy for Graphing $f(x) = \sin x$ and $g(x) = \cos x$ - You must memorize this!

NOTE: To produce graphs of trigonometric functions with a hand-held graphing calculator, be sure to switch to radian mode!!!

1. Graph the *representative picture*.

   - Draw a *Cartesian Coordinate System* showing a horizontal and vertical axis. Most likely you will use an x-axis and a y-axis.

   - Mark off a distance along the x-axis starting at the origin $(0, 0)$ to represent the *period* $2\pi \approx 6.28$. YOU decide how long this distance is going to be in YOUR coordinate system.

   - Divide the *period* into **four equal intervals**.

   - For the beginning and ending point of each interval within a *period*, calculate the y-value and plot the points. This will give you your peaks, valleys, and x-intercepts. See "Observations" above.

     Remember, the beginning and ending point of each interval within a *period* have a certain x-value and given $y = f(x)$ or $y = g(x)$ you can then find the corresponding y-value.

   - Connect the points to form either the *representative picture* of the sine or cosine function.

2. Copy several more cycles in the same manner to the right and left of the *representative picture*.

Transformations of the Graphs of the Sine and Cosine Functions

Previously, we discussed the functions $f(x) = \sin x$ and $g(x) = \cos x$ and how to graph them. Let's call them BASIC FUNCTIONS.

Now we will investigate transformations of these basic functions, specifically of the form

$f(x) = a\sin(bx - c) + d$ and $g(x) = a\cos(bx - c) + d$, where

a, b, c, and ***d*** can take on any value, except
a and ***b*** cannot be equal to ***0***

NOTE: You should have discussed transformations of the graphs of functions in detail in your prerequisite algebra course!

- The value ***a*** indicates a **vertical stretch or shrink** in the graph of the basic functions. This will change the y-values of the basic functions.

  The absolute value of ***a*** in both functions is called the **Amplitude**. The following formula can be used to describe the amplitude

$$|a| = \frac{|max\ \ value\ \ of\ \ sin\ \ or\ \ cos| + |min\ \ value\ \ of\ \ sin\ \ or\ \ cos|}{2}$$

- The value ***b*** indicates a **horizontal stretch or shrink** in the graph of the basic functions. This will affect the size of the period of the basic functions.

  The formula $P = \frac{2\pi}{b}$ describes the period ***P*** of the graph of a trigonometric function.

- The value ***c*** causes a **horizontal shift (phase shift)** in the graph of the basic functions.

  The formula $\frac{c}{b}$ describes the phase shift in the graph of a trigonometric function.

- The value ***d*** causes a **vertical shift** in the graph of the basic functions.

Strategy for Graphing the Transformations $f(x) = a\sin(bx)$ and $g(x) = a\cos(bx)$

NOTE: These are the only transformations we will be graphing by hand in this course.

- Use the strategy as outlined above for the basic functions $f(x) = \sin x$ and $g(x) = \cos x$.
- The only difference might be that you may have to use the the formula $P = \frac{2\pi}{b}$ to calculate the period.

Problem 1:

Given ***g(x) = 2 sin(2x)***, do the following:

a. state the amplitude and **EXACT** period
b. graph the function on the interval $[-2\pi, 2\pi]$
c. find the **EXACT** coordinates of the peaks using the graph
d. find the **EXACT** coordinates of the valleys using the graph
e. find the **EXACT** coordinates of the x-intercepts using the graph
f. find $g(\frac{2}{7})$ rounded to four decimal places

a. state the amplitude and **EXACT** period

The amplitude is ***|2| = 2***.

The period is $\frac{2\pi}{2} = \pi$. This means the representative picture is π units in length.

NOTE: To find an **EXACT** period means that we want to express it in terms of π and not use a decimal approximation such as $\pi \approx 3.14$.

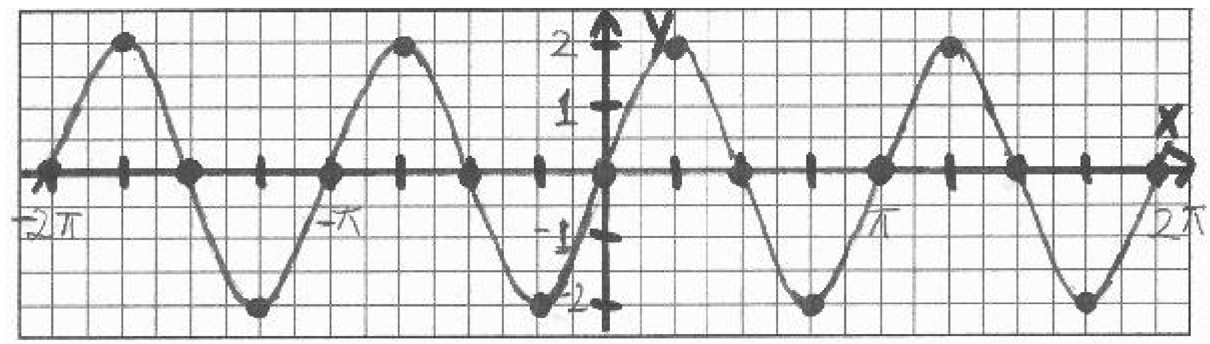
b. graph the function on the interval $[-2\pi, 2\pi]$ as follows:

Graph the representative picture.

- Mark off a distance along the x-axis starting at the origin to represent the period π
- Divide this period into four equal intervals of length $\pi/4 \approx 0.785$.
- For the beginning and ending point of each interval calculate the y-value and plot the points.
- Connect the points to form the representative picture of the sine function.

Copy several more cycles in the same manner to the right and left of the representative picture on the interval $[-2\pi, 2\pi]$.

Please note that the peaks and valleys in the graphs of the sine and cosine functions are U-shaped!



c. find the **EXACT** coordinates of the peaks using the graph

NOTE: To find the **EXACT** coordinates means that we want to express them in terms of π and not use a decimal approximation such as π ≈ ***3.14***.

Remember that the period is divided in four intervals each of length π***/4*** ≈ ***0.785***.

To find the coordinates of the peaks just count the number of intervals from 0 to their x-coordinates.

For example, there is a peak at ***(-7π/4, 2)*** because we must count 7 intervals of length π***/4*** to the **left** of ***0*** to find its x-coordinate.

(-7π/4, 2), (-3π/4, 2), (π/4, 2), (5π/4, 2)

d. find the **EXACT** coordinates of the valleys using the graph

Remember that the period is divided in four intervals each of length π***/4*** ≈ ***0.785***.

To find the coordinates of the valleys just count the number of intervals from 0 to their x-coordinates.

For example, there is a valley at ***(3π/4, -2)*** because we must count 3 intervals of length π***/4*** to the **right** of ***0*** to find its x-coordinate.

(-5π/4, -2), (-π/4, -2), (3π/4, -2), (7π/4, -2)

e. find the **EXACT** coordinates of the x-intercepts using the graph

Remember that the period is divided in four intervals each of length π***/4*** ≈ ***0.785***.

To find the coordinates of the x-intercepts just count the number of intervals from 0 to their x-coordinates.

For example, there is an x-intercept at ***(π, 0)*** because we must count 4 intervals of length ***π/4*** to the **left** of ***0*** to find its x-coordinate.

(-2π, 0), (-3π/2, 0), (-π, 0), (-π/2, 0)

(0, 0), (π/2, 0), (π, 0), (3π/2, 0), (2π, 0)

f. find $g(\frac{2}{7})$ rounded to four decimal places

2	sin	(	2	*	2	/	7	)	Enter

$g(\frac{2}{7}) \approx 1.0817$

Problem 2:

Given $h(x) = -\frac{1}{2}\sin(\pi x)$, do the following:

a. state the amplitude and **EXACT** period
b. graph the function on the interval $[-4, 4]$
c. find the **EXACT** coordinates of the peaks using the graph
d. find the **EXACT** coordinates of the valleys using the graph
e. find the **EXACT** coordinates of the x-intercepts using the graph
f. find $h(\frac{2}{7})$ rounded to four decimal places

a. state the amplitude and **EXACT** period

The amplitude is $\left|-\frac{1}{2}\right| = \frac{1}{2}$.

The period is $\frac{2\pi}{\pi} = 2$. This means the representative picture is ***2*** units in length.

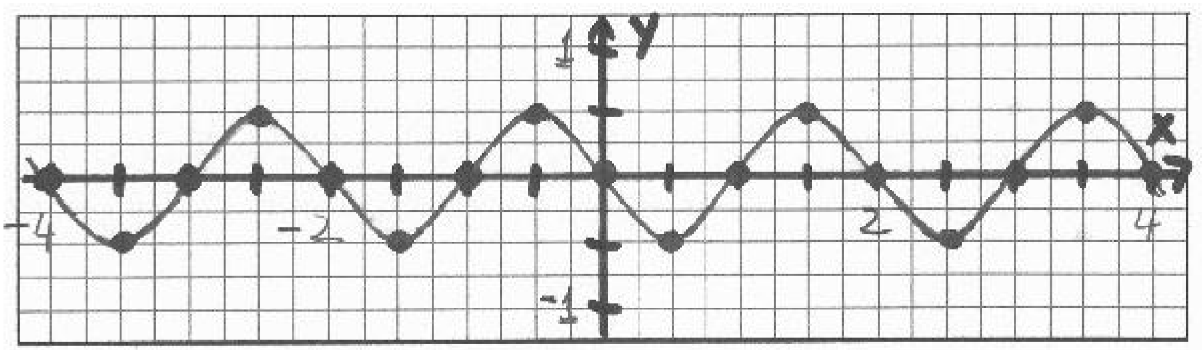
b. graph the function on the interval $[-4, 4]$ as follows:

Graph the representative picture.

- Mark off a distance along the x-axis starting at the origin to represent the period ***2***.
- Divide this period into four equal intervals of length ***0.5***.
- For the beginning and ending point of each interval calculate the y-value and plot the points.
- Connect the points to form the representative picture of the sine function.

Copy several more cycles in the same manner to the right and left of the representative picture on the interval $[-4, 4]$.

Please note that the peaks and valleys in the graphs of the sine and cosine functions are U-shaped!



c. find the **EXACT** coordinates of the peaks using the graph

Remember that each period is divided into four equal intervals each of length ***0.5***!

(-2.5, 0.5), (-0.5, 0.5), (1.5, 0.5), (3.5, 0.5)

d. find the **EXACT** coordinates of the valleys using the graph

Remember that each period is divided into four equal intervals each of length ***0.5***!

(-3.5, -0.5), (-1.5, -0.5), (0.5, -0.5), (2.5, -0.5)

e. find the **EXACT** coordinates of the x-intercepts using the graph

Remember that each period is divided into four equal intervals each of length ***0.5***!

(-4, 0), (-3, 0), (-2, 0), (-1, 0), (0, 0), (1, 0), (2, 0), (3, 0), (4, 0)

f. find $h(\frac{2}{7})$ rounded to four decimal places

(	(-)	1	/	2	)	sin	(	π	*	2	/	7	)	Enter

$h(\frac{2}{7}) \approx$ ***-0.3909***

Problem 3:

Given $f(x) = 2\cos(\frac{x}{2})$, do the following:

a. state the amplitude and **EXACT** period
b. graph the function on the interval $[-8\pi, 8\pi]$
c. find the **EXACT** coordinates of the peaks using the graph
d. find the **EXACT** coordinates of the valleys using the graph
e. find the **EXACT** coordinates of the x-intercepts using the graph
f. find $f(\frac{2}{7})$ rounded to four decimal places

a. state the amplitude and **EXACT** period

The amplitude is $|2| = 2$.

The period is $\frac{2\pi}{\frac{1}{2}} = 4\pi$. This means the representative picture is 4π units in length.

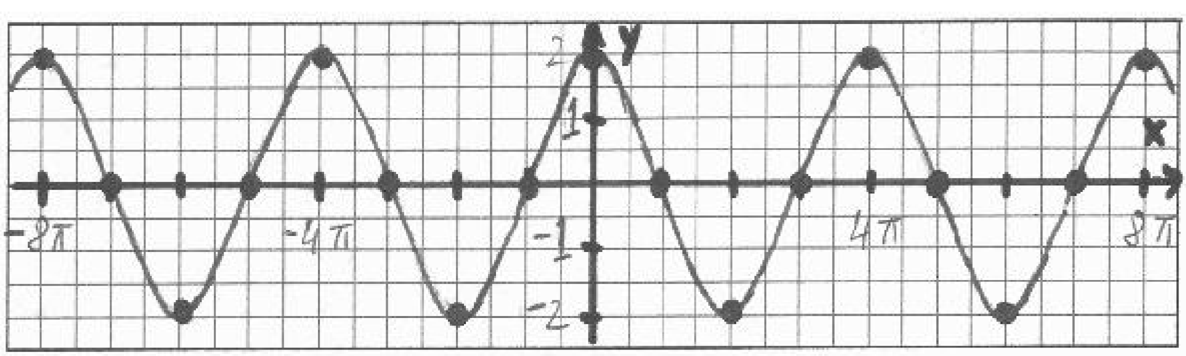
b. graph the function on the interval $[-8\pi, 8\pi]$ as follows:

Graph the representative picture.

- Mark off a distance along the x-axis starting at the origin to represent the period 4π.
- Divide this period into four equal intervals of length π.
- For the beginning and ending point of each interval calculate the y-value and plot the points.
- Connect the points to form the representative picture of the cosine function.

Copy several more cycles in the same manner to the right and left of the representative picture on the interval $[-8\pi, 8\pi]$.

Please note that the peaks and valleys in the graphs of the sine and cosine functions are U-shaped!



c. find the **EXACT** coordinates of the peaks using the graph

Remember that each period is divided into four equal intervals each of length π!

(-8π, 2), (-4π, 2), (0, 2), (4π, 2), (8π, 2)

d. find the **EXACT** coordinates of the valleys using the graph

Remember that each period is divided into four equal intervals each of length π!

(-6π, -2), (-2π, -2), (2π, -2), (6π, -2)

e. find the **EXACT** coordinates of the x-intercepts using the graph

Remember that each period is divided into four equal intervals each of length π!

(-7π, 0), (-5π, 0), (-3π, 0), (-π, 0)

(π, 0), (3π, 0), (5π, 0), (7π, 0)

f. find $f(\frac{2}{7})$ rounded to four decimal places

2	***cos***	***(***	***2***	***/***	***7***	***/***	***2***	***)***	***Enter***

$f(\frac{2}{7}) \approx$ ***1.9796***

Problem 4:

Given $p(x) = -\frac{1}{2}\cos\left(\frac{\pi x}{2}\right)$, do the following:

a. state the amplitude and **EXACT** period
b. graph the function on the interval $[-8,8]$
c. find the **EXACT** coordinates of the peaks using the graph
d. find the **EXACT** coordinates of the valleys using the graph
e. find the **EXACT** coordinates of the x-intercepts using the graph
f. find $p(\frac{2}{7})$ rounded to four decimal places

a. state the amplitude and **EXACT** period

The amplitude is $\left|-\frac{1}{2}\right| = \frac{1}{2}$. The negative sign reflects the function about the x-axis.

The period is $\dfrac{2\pi}{\frac{\pi}{2}} = 4$. This means the representative picture is 4 units in length.

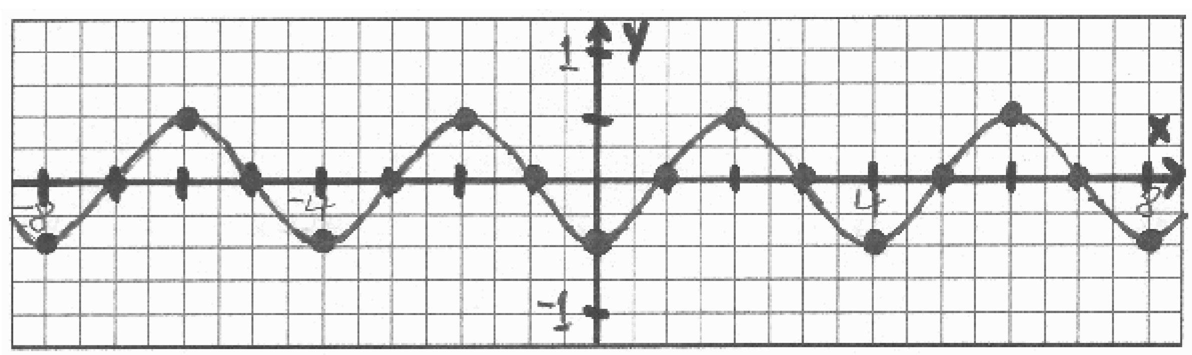
b. graph the function on the interval $[-8,8]$ as follows:

Graph the representative picture.

- Mark off a distance along the x-axis starting at the origin to represent the period 4.
- Divide this period into four equal intervals of length 1.
- For the beginning and ending point of each interval calculate the y-value and plot the points.
- Connect the points to form the representative picture of the cosine function.

Copy several more cycles in the same manner to the right and left of the representative picture on the interval $[-8,8]$.

Please note that the peaks and valleys in the graphs of the sine and cosine functions are U-shaped!



c. find the **EXACT** coordinates of the peaks using the graph

Remember that each period is divided into four equal intervals each of length 1!

(-6, 0.5), (-2, 0.5), (2, 0.5), (6, 0.5)

d. find the **EXACT** coordinates of the valleys using the graph

Remember that each period is divided into four equal intervals each of length 1!

(-8, -0.5), (-4, -0.5), (0, -0.5), (4, -0.5), (8, -0.5)

e. find the **EXACT** coordinates of the x-intercepts using the graph

Remember that each period is divided into four equal intervals each of length ***1***!

(-7, 0), (-5, 0), (-3, 0), (-1, 0), (1, 0), (3, 0), (5, 0), (7, 0)

f. find $p(\frac{2}{7})$ rounded to four decimal places

(	***(-)***	***1***	/	***2***	***)***	***cos***	***(***	***(***	π	/	***2***	***)***	***(***	***2***	/	***7***	***)***	***)***	***Enter***

$p(\frac{2}{7}) \approx$ ***-0.4505***