DETAILED SOLUTIONS AND CONCEPTS - RATIONAL FUNCTIONS

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PLEASE NOTE THAT YOU CANNOT ALWAYS USE A CALCULATOR ON THE ACCUPLACER - COLLEGE-LEVEL MATHEMATICS TEST! YOU MUST BE ABLE TO DO SOME PROBLEMS WITHOUT A CALCULATOR!

Definition of Rational Functions

A rational function is one that can be written in the form $R(x)=\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions, and $q(x)$ is not equal to 0.

The domain MUST exclude any numbers that will make the denominator equal to 0.

By definition, all polynomial functions are also rational functions. For example, if we write the polynomial (quadratic) function

$f(x)=\frac{2}{7}x^2-\frac{5}{7}$ as $f(x)=\frac{2x^2-5}{7}$,

it is now in the form $R(x)=\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions.

Characteristics of the Graphs of Rational Functions (excluding polynomial functions)

Rational functions do not have one single standard graph. Infinitely many different graphs are possible. Following are the characteristics of rational functions that are not polynomial functions.

- The graph consists of SMOOTH curves with rounded turns.
- Since the denominator of a rational function cannot take on the value of 0, in SOME functions numbers might have to be excluded from the domain, which will result in a graph that is NOT continuous at these numbers.
- The numbers that are excluded from the domain of some rational functions either produce *holes* and/or *vertical asymptotes*. The *vertical asymptotes* are invisible vertical lines that divide the graph into two or more branches.
- The branches of the graph continue to move toward/away from their vertical asymptotes at a steady pace. However, they will never touch them.

- The graphs of some functions may also contain *horizontal asymptotes*, which are invisible horizontal lines.
- Other graphs may have *oblique asymptotes*, which are invisible lines that are neither vertical nor horizontal.
- Some graphs will touch or cross their horizontal or oblique asymptotes at distinct points. However, in general, the branches of the graph will continue to move toward/away from the horizontal and oblique asymptotes at a steady pace.
- There may neither be an x- or a y-intercept. If there is a y-intercept, there is only one. Some graphs may have infinitely many x-intercepts.

Definition of the Vertical Asymptote

In a function, the line $x = a$ is a *vertical asymptote* of the graph, if the **absolute** y-values become infinitely large as the x-values get closer and closer to the number a from the right and/or from the left.

NOTE: The number a is NOT in the domain of the function.

Definition of the Horizontal Asymptote

In a function, the line $y = b$ is a *horizontal asymptote* of the graph, if the y-values get closer and closer to the number b as the **absolute** x-values become infinitely large.

NOTE: The number b is NOT in the range of the function.

Definition of Holes

In a function, the point (c, d) is a *hole* in the graph, if the y-values get closer and closer to the number d as the x-value gets closer and closer to the number c.

Strategy for finding the Equations of Asymptotes

- **Vertical Asymptotes**

a. If a function is **reduced to lowest terms** (greatest common factor of numerator and denominator is 1), then set the entire denominator equal to 0 and solve for x. These solutions are the equations of the *vertical asymptote(s)*.

b. If a function is **NOT** reduced to lowest terms, then set the denominator equal to 0 **that remains AFTER "pretend-canceling" the *like* factors.**

Note that we say "pretend-canceling" because we are not allowed to actually reduce the function since this would change the domain. As soon as the domain is different we have a different function!!!

- **Horizontal Asymptotes**

Note that a *horizontal asymptote* is a horizontal line, and equations of horizontal lines are of the form $y = b$, where b is any real number.

In a rational function $R(x)=\frac{p(x)}{q(x)}$, where ***p(x)*** and ***q(x)*** are polynomial functions, you can find the equations of the *horizontal asymptotes* as follows:

- If the degree of ***p(x)*** **is less than** the degree of ***q(x)***, the *horizontal asymptote* is the x-axis. Its equation is ***y = 0***.
- If the degree of ***p(x)*** **is equal to** the degree of ***q(x)***, the horizontal line

$$y=\frac{\text{leading coefficient of } p(x)}{\text{leading coefficient of } q(x)}$$ is the *horizontal asymptote*.

- If the degree of ***p(x)*** **is greater than** the degree of ***q(x)***, there is no *horizontal asymptote*.

- **Oblique Asymptotes**

If the degree of the numerator of a function is exactly one greater than that of the denominator, the graph might have an *oblique asymptote*. The equation of an oblique asymptote is the quotient resulting from polynomial long division when dividing the denominator into the numerator. However, to actually get an oblique asymptote the remainder CANNOT be ***0***.

Strategy for finding the Coordinates of Holes

Please note that holes in the graph of a rational function can only be produced by factors that are common to the numerator and the denominator (see exception in Example 10)!

x-coordinate:

It is found by setting the factors equal to ***0*** that are common to the numerator and the denominator of the function. If the degree of these factors is greater in the denominator than in the numerator, they will NOT produce a hole.

y-coordinate

First, we MUST create a **new function** by reducing the original function. Remember that it has to be a "new" function because of the domain changes due to canceling of *like* factors (see strategy for finding *vertical asymptotes*). Then, to find the y-coordinate of the hole, we simply replace ***x*** in the **new function** with the x-coordinate of the hole.

Problem 1:

Given the rational function $f(x) = \frac{1}{x}$, which is also called the **Reciprocal Function**, find the following:

a. the Domain in Set-Builder Notation
b. the Equation of any Vertical Asymptotes
c. the Equation of any Horizontal Asymptotes
d. the Equations of any Oblique Asymptotes
e. the Coordinates of any Holes

- Domain:

  You must exclude from the domain any number that would make the denominator equal to 0 and, thus, the y-value undefined.

  Its domain is $(-\infty, 0) \cup (0, \infty)$.

- Equation of the *Vertical Asymptote:*

  Since the function is reduced to lowest terms, we set the denominator equal to 0 and solve.

  In our case, $x = 0$ is the equation of the *vertical asymptote*

- Equation of the *Horizontal Asymptote*:

  Since the degree of the numerator is less than the degree of the denominator, the equation of the *horizontal asymptote* is $y = 0$, which is the x-axis.

- Equation of the *Oblique Asymptote:*

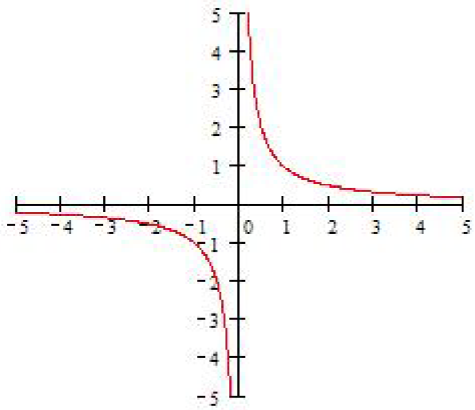
  Since the degree of the numerator is NOT exactly one greater than that of the denominator, the graph does not have any *oblique asymptote*.

- Coordinates of any Holes:

  The original function is reduced to lowest terms, therefore, we do not need to worry about holes.

Below is the graph of the function.

Please note that the branches of rational functions have SMOOTH turns. They are never parallel to their asymptotes, but move toward them at a steady pace.

	Please note that the x-axis is a *horizontal asymptote* and the y-axis is the *vertical asymptote.*

Problem 2:

Given the rational function $f(x) = \dfrac{4x}{2x^2 + 1}$, find the following:

a. the Domain in Interval Notation
b. the Equation of any Vertical Asymptotes
c. the Equation of any Horizontal Asymptotes
d. the Equations of any Oblique Asymptotes
e. the Coordinates of any Holes

- Domain:

  You must exclude from the domain any number that would make the denominator equal to 0 and, thus, the y-value undefined.

  In "matherese" we have to write this as follows:

  $$2x^2 + 1 = 0$$
  $$x^2 = -\tfrac{1}{2}$$
  $$x = \pm\sqrt{-\tfrac{1}{2}} = \pm\tfrac{i\sqrt{2}}{2}$$

  Since only imaginary numbers would make the denominator equal to 0, the domain of this function consists of ***All Real Numbers*** or $(-\infty, \infty)$ in *Interval Notation.*

- Equation of the *Vertical Asymptote*:

  Since the function is reduced to lowest terms, we set the denominator equal to $\boldsymbol{0}$.

  $$2x^2 + 1 = 0$$
  $$x^2 = -\tfrac{1}{2}$$
  $$x = \pm\sqrt{-\tfrac{1}{2}} = \pm\tfrac{i\sqrt{2}}{2}$$

  Since the values of $\boldsymbol{x}$ are imaginary numbers, there are **NO** *vertical asymptotes*.

- Equation of the *Horizontal Asymptote:*

  Since the degree of the numerator is less than the degree of the denominator, the equation of the *horizontal asymptote* is $\boldsymbol{y = 0}$, which is the x-axis.

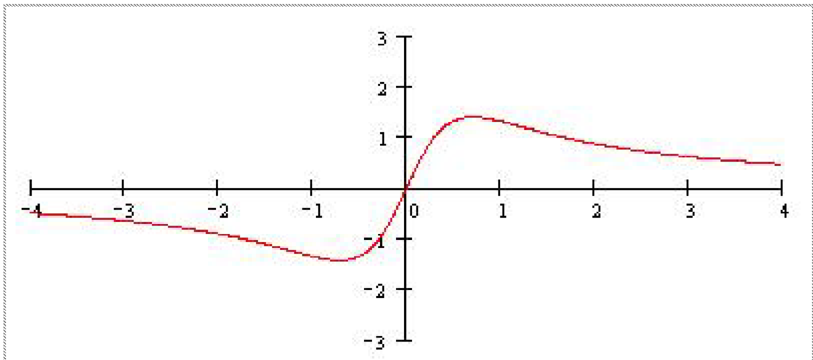
- Equation of the *Oblique Asymptote:*

  Since the degree of the numerator is NOT exactly one greater than that of the denominator, the graph does not have any *oblique asymptote*.

- Coordinates of any Holes:

  The original function is reduced to lowest terms, therefore, we do not need to worry about holes.

Below is the graph of the function.



Please note that the x-axis is a *horizontal asymptote*. There are no *vertical asymptotes*! Remember that graphs may cross/touch their *horizontal asymptotes*!

Problem 3:

Given the rational function $g(x) = \dfrac{4x^2}{x^2 - 1}$, find the following:

a. the Domain in Set-Builder Notation
b. the Equation of any Vertical Asymptotes
c. the Equation of any Horizontal Asymptotes
d. the Equations of any Oblique Asymptotes
e. the Coordinates of any Holes

- Domain:

  You must exclude from the domain any number that would make the denominator equal to 0 and, thus, the y-value undefined.

  $$x^2 - 1 = 0$$
  $$x^2 = 1$$
  $$x = \pm\sqrt{1} = \pm 1$$

  The domain consists of ***All Real Numbers except -1 and 1*** or $\{x \mid x \neq -1, x \neq 1\}$.

  Note: In the case of rational functions, the domain is usually expressed in *Set-Builder Notation* for simplicity's sake.

  We could have used *Interval Notation*, however, it is usually quite cumbersome when having to exclude more than one number from the domain. See for yourself:

  $$(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$$

- Equation of the *Vertical Asymptote*:

  Since the function is reduced to lowest terms, we set the denominator equal to 0

  $$x^2 - 1 = 0$$

  $$x^2 = 1$$

  $$x = \pm 1$$

  This rational function has two *vertical asymptotes*. Their equations are $x = -1$ and $x = 1$.

- Equation of the *Horizontal Asymptote:*

  Since the degree of the numerator is equal to the degree of the denominator, the *horizontal asymptote* is the horizontal line

  $y = \frac{4}{1}$, where 4 is the leading coefficient of the polynomial in the numerator, and 1 is the leading coefficient of the polynomial in the denominator.

  The equation of the *horizontal asymptote* is $y = 4$.

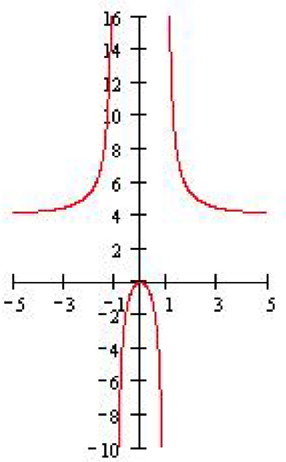
- Equation of the *Oblique Asymptote:*

  Since the degree of the numerator is NOT exactly one greater than that of the denominator, the graph does not have any *oblique asymptote*.

- Coordinates of any Holes:

  The original function is reduced to lowest terms, therefore, we do not need to worry about holes.

Below is the graph of the function.



There is a *horizontal asymptote* at *y = 4* and two *vertical asymptotes* at $x = -1$ and $x = 1$. Please note that asymptotes are invisible lines and NO graphing program places lines into a graph to represent asymptotes.

Problem 4:

Given the rational function $h(x)=\frac{4x^5+11x^2}{2x^2+1}$, find the following:

a. the Domain in Interval Notation
b. the Equation of any Vertical Asymptotes
c. the Equation of any Horizontal Asymptotes
d. the Equations of any Oblique Asymptotes
e. the Coordinates of any Holes

- Domain:

  You must exclude from the domain any number that would make the denominator equal to 0 and, thus, the y-value undefined.

  $$2x^2+1=0$$
  $$x^2=-\tfrac{1}{2}$$
  $$x=\pm\sqrt{-\tfrac{1}{2}}=\pm\tfrac{i\sqrt{2}}{2}$$

  Since only imaginary numbers would make the denominator equal to 0, the domain of this function consists of ***All Real Numbers*** or $(-\infty,\infty)$ in *Interval Notation*.

- Equation of the *Vertical Asymptote*:

  Since the function is reduced to lowest terms, we set the denominator equal to 0

  $$2x^2+1=0$$
  $$x^2=-\tfrac{1}{2}$$
  $$x=\pm\sqrt{-\tfrac{1}{2}}=\pm\tfrac{i\sqrt{2}}{2}$$

  Since the values of x are imaginary numbers, there are NO *vertical asymptotes*.

- Equation of the *Horizontal Asymptote*:

  Since the degree of the numerator is larger than the degree of the denominator, there are NO *horizontal asymptotes*.

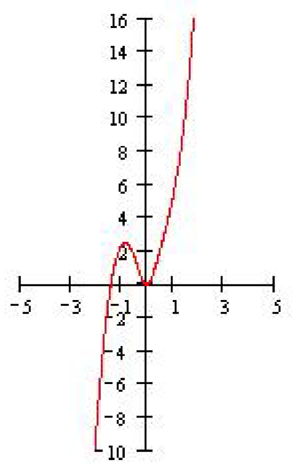
- Equation of the *Oblique Asymptote:*

  Since the degree of the numerator is NOT exactly one greater than that of the denominator, the graph does not have any *oblique asymptote*.

- Coordinates of any Holes:

  The original function is reduced to lowest terms, therefore, we do not need to worry about holes.

Below is the graph of the function.

	Please note that there are no asymptotes!

Problem 5:

Given the rational function $f(x)=\dfrac{2x^2+9x-6}{x^2+2x-3}$, find the following:

a. the Domain in Set-Builder Notation
b. the Equation of any Vertical Asymptotes
c. the Equation of any Horizontal Asymptotes
d. the Equations of any Oblique Asymptotes
e. the Coordinates of any Holes

- Domain:

  You must exclude from the domain any number that would make the denominator equal to 0 and, thus, the y-value undefined.

  $$x^2+2x-3=0$$
  $$(x+3)(x-1)=0$$

  Then $x=-3$ or $x=1$

  The domain consists of ***All Real Numbers except -3 and 1*** or $\{x \mid x \neq -3, x \neq 1\}$.

- Equation of the *Vertical Asymptote*:

  Since the function is reduced to lowest terms, we set the denominator equal to 0

$$f(x) = \frac{2x^2 + 9x - 6}{(x+3)(x-1)}$$ Please note that the numerator cannot be factored relative to the integers!

$$x^2 + 2x - 3 = 0$$
$$(x+3)(x-1) = 0$$

Then $x = -3$ or $x = 1$

This rational function has two *vertical asymptotes*. Their equations are $x = -3$ and $x = 1$.

- Equation of the *Horizontal Asymptote*:

  Since the degree of the numerator is equal to the degree of the denominator, the *horizontal asymptote* is the horizontal line

  $y = \frac{2}{1}$, where 2 is the leading coefficient of the polynomial in the numerator, and 1 is the leading coefficient of the polynomial in the denominator.

  The equation of the *horizontal asymptote* is $y = 2$.

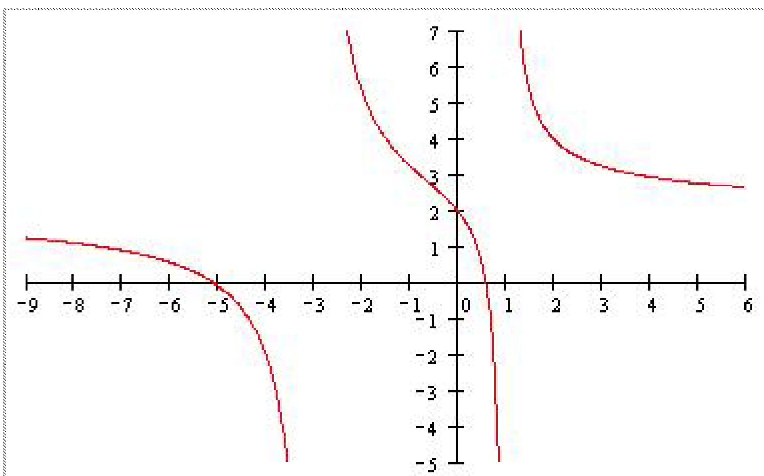
- Equation of the *Oblique Asymptote:*

  Since the degree of the numerator is NOT exactly one greater than that of the denominator, the graph does not have any *oblique asymptote*.

- Coordinates of any Holes:

  The original function is reduced to lowest terms, therefore, we do not need to worry about holes.

Below is the graph of the function.



There is a *horizontal asymptote* at $y = 2$ and two *vertical asymptotes* at $x = -3$ and $x = 1$. Remember that graphs may cross/touch their *horizontal asymptotes*!

Problem 6:

Given the rational function $f(x) = \frac{4x^2}{2x^2+1}$, find the following:

a. the Domain in Interval Notation
b. the Equation of any Vertical Asymptotes
c. the Equation of any Horizontal Asymptotes
d. the Equations of any Oblique Asymptotes
e. the Coordinates of any Holes

- Domain:

  You must exclude from the domain any number that would make the denominator equal to 0 and, thus, the y-value undefined.

  $$2x^2 + 1 = 0$$
  $$x^2 = -\tfrac{1}{2}$$
  $$x = \pm\sqrt{-\tfrac{1}{2}} = \pm\tfrac{i\sqrt{2}}{2}$$

  Since only imaginary numbers would make the denominator equal to 0, the domain of this function consists of ***All Real Numbers*** or $(-\infty, \infty)$ in *Interval Notation*.

- Equations of the *Vertical Asymptote:*

  Since the function is reduced to lowest terms, we set the denominator equal to 0

  $$2x^2 + 1 = 0$$
  $$x^2 = -\tfrac{1}{2}$$
  $$x = \pm\sqrt{-\tfrac{1}{2}} = \pm\tfrac{i\sqrt{2}}{2}$$

  Since the values of x are imaginary numbers, there are **NO** *vertical asymptotes*.

- Equation of the *Horizontal Asymptote*:

  Since the degree of the numerator is equal to the degree of the denominator, the *horizontal asymptote* is the horizontal line

  $y = \frac{4}{2}$, where 4 is the leading coefficient of the polynomial in the numerator, and 2 is the leading coefficient of the polynomial in the denominator.

  The equation of the *horizontal asymptote* is $y = 2$.

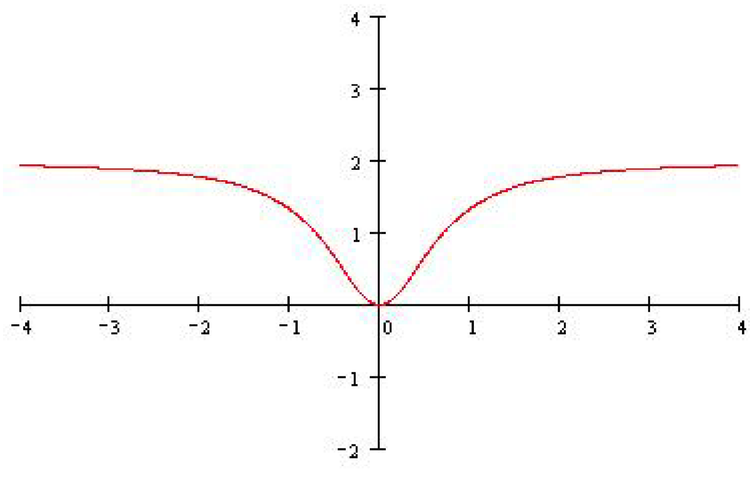
- Equation of the *Oblique Asymptote:*

  Since the degree of the numerator is NOT exactly one greater than that of the denominator, the graph does not have any *oblique asymptote*.

- Coordinates of any Holes:

  The original function is reduced to lowest terms, therefore, we do not need to worry about holes.

Below is the graph of the function.



There is a *horizontal asymptote* at $y = 2$ and no *vertical asymptotes*.

Problem 7:

Given the rational function $f(x) = \dfrac{-x^2 - 4x}{(x+2)^2}$, find the following:

a. the Domain in Set-Builder Notation
b. the Equation of any Vertical Asymptotes
c. the Equation of any Horizontal Asymptotes
d. the Equations of any Oblique Asymptotes
e. the Coordinates of any Holes

- Domain:

  You must exclude from the domain any number that would make the denominator equal to 0 and, thus, the y-value undefined.

  $$(x+2)(x+2) = 0$$

  Then $x = -2$

  The domain consists of ***All Real Numbers except -2*** or $\{x \mid x \neq -2\}$.

- Equation of the *Vertical Asymptote*:

  Since the function is reduced to lowest terms, we set the denominator equal to ***0*** and solve.

  $$f(x) = \frac{-x(x+4)}{(x+2)(x+2)}$$

  $$(x+2)(x+2) = 0$$

  Then $x = -2$

  This rational function has one *vertical asymptote*. Its equation is $x = -2$.

- Equation of the *Horizontal Asymptote:*

  Since the degree of the numerator is equal to the degree of the denominator, the *horizontal asymptote* is the horizontal line

  $y = \frac{-1}{1}$, where ***-1*** is the leading coefficient of the polynomial in the numerator, and ***1*** is the leading coefficient of the polynomial in the denominator.

  The equation of the *horizontal asymptote* is $y = -1$.

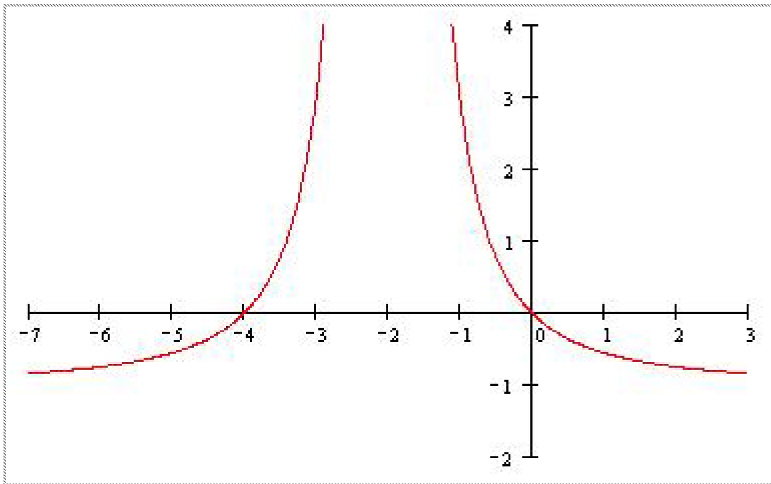
- Equation of the *Oblique Asymptote:*

  Since the degree of the numerator is NOT exactly one greater than that of the denominator, the graph does not have any *oblique asymptote*.

- Coordinates of any Holes:

  The original function is reduced to lowest terms, therefore, we do not need to worry about holes.

Below is the graph of the function.



There is a *horizontal asymptote* at $y = -1$ and a *vertical asymptote* at $x = -2$.

Problem 8:

Given the rational function $p(x) = \dfrac{x^2 - 3}{2x - 4}$, find the following:

a. the Domain in Set-Builder Notation
b. the Equation of any Vertical Asymptotes
c. the Equation of any Horizontal Asymptotes
d. the Equations of any Oblique Asymptotes
e. the Coordinates of any Holes

- Domain:

  You must exclude from the domain any number that would make the denominator equal to 0 and, thus, the y-value undefined.

  $2x - 4 = 0$

  $x = 2$

  The domain consists of ***All Real Numbers except 2*** or $\{x \mid x \neq 2\}$.

- Equation of the *Vertical Asymptote*:

  Since the function is reduced to lowest terms, we set the denominator equal to 0 and solve.

  $$p(x) = \frac{x^2 - 3}{2x - 4}$$

  $2x - 4 = 0$ and $x = 2$

  This rational function has one *vertical asymptote*. Its equation is $x = 2$.

- Equation of the *Horizontal Asymptote*:

  The degree of the numerator is larger than the degree of the denominator. This means that there are NO *horizontal asymptotes*. Since the degree of the numerator is exactly one larger than the degree of the denominator, this graph will have *oblique asymptotes*.

- Equation of the *Oblique Asymptote:*

  We determine the equation of this asymptote by finding the quotient resulting from polynomial long division when dividing the denominator into the numerator.

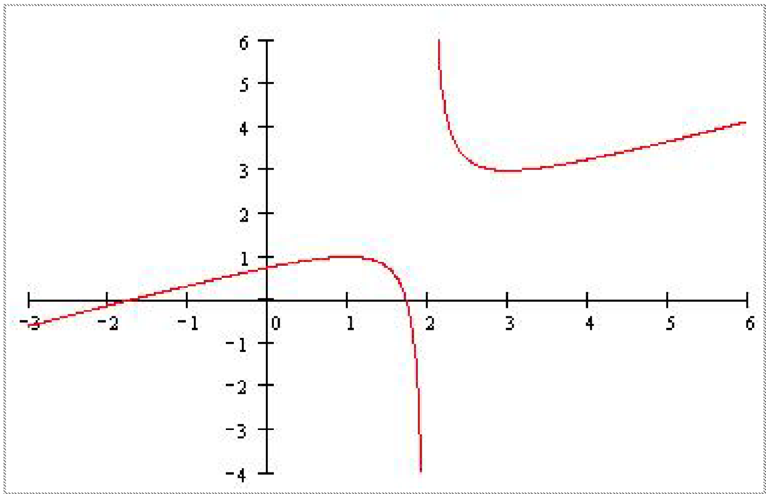
$$\begin{array}{r} \tfrac{1}{2}x + 1 \\ 2x-4\,\overline{)\,x^2 + 0x - 3} \\ -(x^2 - 2x) \\ \hline 2x - 3 \\ -(2x-4) \\ \hline 1 \end{array}$$

  Note that the function can now be written as $p(x) = \frac{1}{2}x + 1 + \dfrac{1}{2x-4}$, where the equation $y = \frac{1}{2}x + 1$ represents the *oblique asymptote*.

- Coordinates of any Holes:

  The original function is reduced to lowest terms, therefore, we do not need to worry about holes.

Below is the graph of the function.



There is a *vertical asymptote* at $x = 2$ and an *oblique asymptote* at $y = \frac{1}{2}x + 1$.

Problem 9:

Given the rational function $f(x) = \dfrac{x+2}{x^2 - 4}$, find the following:

a. the Domain in Set-Builder Notation
b. the Equation of any Vertical Asymptotes
c. the Equation of any Horizontal Asymptotes
d. the Equations of any Oblique Asymptotes
e. the Coordinates of any Holes

- Domain:

  You must exclude from the domain any number that would make the denominator equal to 0 and, thus, the y-value undefined.

  $$x^2 - 4 = 0$$
  $$x^2 = 4$$
  $$x = \pm\sqrt{4} = \pm 2$$

  The domain consists of ***All Real Numbers except -2 and 2*** or $\{x \mid x \neq -2, x \neq 2\}$.

- Equation of the *Vertical Asymptote*:

  Please note that this function is not reduced to lowest terms. That is,

  $$f(x) = \frac{x+2}{x^2 - 4} = \frac{x+2}{(x+2)(x-2)}$$

  In this case, setting the denominator equal to 0 will NOT result in two *vertical asymptotes*.

NOTE: You **cannot** reduce the function f because this would change its domain, in which case we would have a different function.

We find the equation of the *vertical asymptote* by setting equal to 0 ONLY that factor in the denominator that **DOES NOT** also appear in the numerator.

$x - 2 = 0$

$x = 2$

The *vertical asymptote* is $x = 2$.

Note that the graph of this function MUST have a hole at the other number that is excluded from the domain, namely $x = -2$!

- Equation of the *Horizontal Asymptote*:

  Since the degree of the numerator is less than the degree of the denominator, the *horizontal asymptote* is the x-axis.

  The equation of the horizontal asymptote is $y = 0$, which is the x-axis.

- Equation of the *Oblique Asymptote:*

  Since the degree of the numerator is NOT exactly one greater than that of the denominator, the graph does not have any *oblique asymptote*.

- Coordinates of any Holes:

  Since the function is NOT reduced to lowest terms its graph must have one or more holes.

x-coordinate:

It is found by setting equal to 0 the factor that appears both in the numerator and the denominator of the function

$$f(x) = \frac{x+2}{x^2-4} = \frac{x+2}{(x+2)(x-2)}$$

In our case,

$x + 2 = 0$

$x = -2$

y-coordinate:

First, we MUST create a **new function** by reducing the original function. That is,

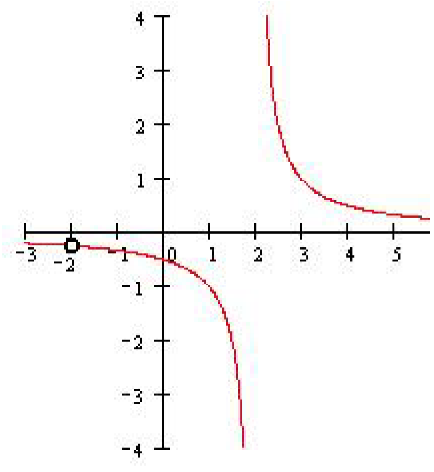
$g(x) = \frac{1}{x-2}$ with domain $\{x \mid x \neq 2\}$

To find the y-coordinate of the hole, we simply replace x in the reduced function g with the x-coordinate of the hole

$$g(-2) = \frac{1}{-2-2} = -\frac{1}{4}$$

Finally, the coordinates of the hole are $(-2, -\frac{1}{4})$.

Below is the graph of the function.



Please note that the x-axis is a *horizontal asymptote*. There is also a *vertical asymptote* at $x = 2$. Please observe the hole in the graph at $(-2, -\frac{1}{4})$.

Problem 10:

Given the rational function $f(x) = \frac{x^2 + 2x - 3}{2x^2 - 12x + 10}$, find the following:

a. the Domain in Set-Builder Notation
b. the Equation of any Vertical Asymptotes
c. the Equation of any Horizontal Asymptotes
d. the Equations of any Oblique Asymptotes
e. the Coordinates of any Holes

- Domain:

You must exclude from the domain any number that would make the denominator equal to 0 and, thus, the y-value undefined.

$$2x^2 - 12x + 10 = 0$$
$$2(x^2 - 6x + 5) = 0$$
$$2(x - 1)(x - 5) = 0$$

Then $x = 1$ or $x = 5$

The domain consists of ***All Real Numbers except 1 and 5*** or $\{x \mid x \neq 1, x \neq 5\}$.

- Equation of the *Vertical Asymptote*:

  Please note that this function is not reduced to lowest terms. That is,

$$f(x) = \frac{x^2 + 2x - 3}{2x^2 - 12x + 10} = \frac{(x-1)(x+3)}{2(x-1)(x-5)}$$

  In this case, setting the denominator equal to 0 will NOT result in two *vertical asymptotes*.

  We find the equation of the *vertical asymptote* by setting equal to 0 ONLY that factor in the denominator that **DOES NOT** also appear in the numerator.

  $x - 5 = 0$

  $x = 5$

  The *vertical asymptote* is $x = 5$.

  Note that the graph of this function MUST have a hole at the other number that is excluded from the domain, namely $x = 1$!

- Equation of the *Horizontal Asymptote*:

  Since the degree of the numerator is equal to the degree of the denominator, the *horizontal asymptote* is the horizontal line

  $y = \frac{1}{2}$, where 1 is the leading coefficient of the polynomial in the numerator, and 2 is the leading coefficient of the polynomial in the denominator.

  The equation of the *horizontal asymptote* is $y = \frac{1}{2}$.

- Equation of the *Oblique Asymptote:*

  Since the degree of the numerator is NOT exactly one greater than that of the denominator, the graph does not have any *oblique asymptote*.

- Coordinates of any Holes:

  Since the function is NOT reduced to lowest terms its graph must have one or more holes.

  x-coordinate:

  It is found by setting equal to $\boldsymbol{0}$ the factor that appears both in the numerator and the denominator of the function

  $$f(x) = \frac{x^2 + 2x - 3}{2x^2 - 12x + 10} = \frac{(x-1)(x+3)}{2(x-1)(x-5)}$$

  In our case,

  $\boldsymbol{x - 1 = 0}$

  $\boldsymbol{x = 1}$

  y-coordinate:

  First, we MUST create a **new function** by reducing the original function. That is,

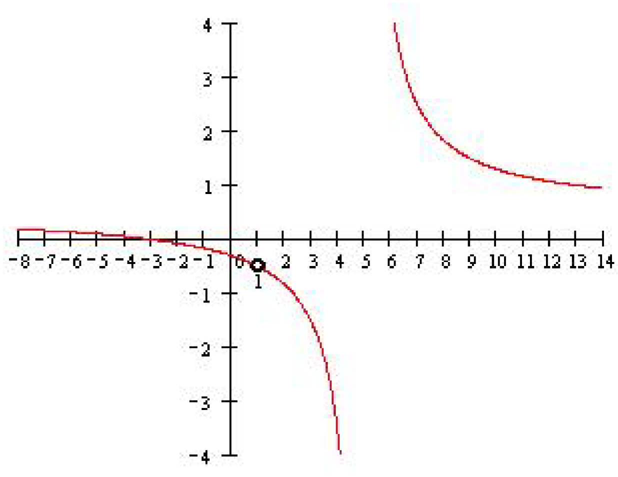
  $$g(x) = \frac{x+3}{2(x-5)}$$ with domain $\{x \mid x \neq 5\}$

  To find the y-coordinate of the hole, we simply replace $\boldsymbol{x}$ in the reduced function $\boldsymbol{g}$ with the x-coordinate of the hole

  $$g(1) = \frac{1+3}{2(1-5)} = \frac{4}{-8} = -\frac{1}{2}$$

  Finally, the coordinates of the hole are $(1, -\frac{1}{2})$.

Below is the graph of the function.

	There is a *horizontal asymptote* at $y = \frac{1}{2}$ and a *vertical asymptote* at $x = 5$. Please observe the hole in the graph at $(1, -\frac{1}{2})$.

Problem 11:

Given the rational function $k(x) = \dfrac{x-1}{x^2-1}$, find the following:

a. the Domain in Set-Builder Notation
b. the Equation of any Vertical Asymptotes
c. the Equation of any Horizontal Asymptotes
d. the Equations of any Oblique Asymptotes
e. the Coordinates of any Holes

- Domain:

  You must exclude from the domain any number that would make the denominator equal to 0 and, thus, the y-value undefined.

  $$x^2 - 1 = 0$$
  $$(x-1)(x+1) = 0$$

  then $x = 1$ and $x = -1$

  Its domain consists of ***All Real Numbers except -1 and 1*** or $\{x \mid x \neq -1, x \neq 1\}$.

- Equation of the *Vertical Asymptote*:

  Please note that this function is **NOT** reduced to lowest terms. That is,

  $$k(x) = \frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)}$$

  In this case, setting the denominator equal to 0 will **NOT** result in two *vertical asymptotes.*

We find the *vertical asymptote* by setting equal to $\boldsymbol{0}$ the factor in the denominator that **DOES NOT** also appear in the numerator.

$x + 1 = 0$

$x = -1$

The *vertical asymptote* is $x = -1$.

- Equation of the *Horizontal Asymptote:*

  $y = 0$ (degree of numerator less than degree of denominator). This is the x-axis.

- Equation of the *Oblique Asymptote:*

  Since the degree of the numerator is NOT exactly one greater than that of the denominator, the graph does not have any *oblique asymptote*.

- Coordinates of any Holes:

  Since the function is NOT reduced to lowest terms its graph must have one or more holes.

x-coordinate

It is found by setting equal to $\boldsymbol{0}$ the factor that appears both in the numerator and the denominator of the function

$$k(x) = \frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)}$$

In our case,

$x - 1 = 0$

$x = 1$

y-coordinate

First, we MUST create a **new function** by reducing the original function. That is,

$$g(x) = \frac{1}{x+1} \text{ with domain } \{x \mid x \neq -1\}$$

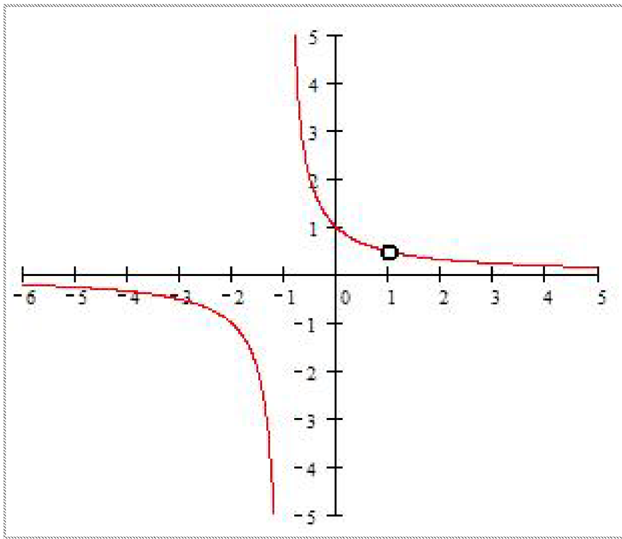
To find the y-coordinate of the hole, we simply replace x in the reduced function g with the x-coordinate of the hole.

$$g(1) = \frac{1}{1+1} = \frac{1}{2}$$

Finally, the coordinates of the hole are $(1, \frac{1}{2})$.

Below is the graph of the function.

NOTE: The graphs of the functions k and g are almost identical. Both have a *Vertical Asymptote* at $x = -1$. The only difference between them is that the function k has a **hole** at $(1, \frac{1}{2})$.



There is a *horizontal asymptote* at $y = 0$ and a *vertical asymptote* at $x = -1$. Please observe the hole in the graph at $(1, \frac{1}{2})$.

Problem 12:

Given the rational function $h(x) = \frac{x^2 - 1}{x - 1}$, find the following:

a. the Domain in Set-Builder Notation
b. the Equation of any Vertical Asymptotes
c. the Equation of any Horizontal Asymptotes
d. the Equations of any Oblique Asymptotes
e. the Coordinates of any Holes

- Domain:

  You must exclude from the domain any number that would make the denominator equal to 0 and, thus, the y-value undefined.

  $x - 1 = 0$

  $x = 1$

  Its domain consists of ***All Real Numbers except 1*** or $\{x \mid x \neq 1\}$.

- Equation of the *Vertical Asymptote:*

  Please note that this function is NOT reduced to lowest terms. That is,

  $$h(x) = \frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x-1}$$

  In this case, setting the denominator equal to ***0*** will NOT result in a *vertical asymptote*.

  We find the *vertical asymptote* by setting equal to ***0*** the factor in the denominator that **DOES NOT** also appear in the numerator.

  This is not possible in our case, therefore, there is ***NO*** *vertical asymptote*.

- Equation of the *Horizontal Asymptote:*

  NONE (degree of numerator greater than degree of denominator)

- Equation of the *Oblique Asymptote:*

  Since the degree of the numerator is NOT exactly one greater than that of the denominator, the graph does not have any *oblique asymptote*.

- Coordinates of the Hole:

  Since the function is NOT reduced to lowest terms its graph must have one or more holes.

  x-coordinate

  It is found by setting equal to ***0*** the factor that appears both in the numerator and the denominator of the function

  $$h(x) = \frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x-1}$$

  In our case,

  $x - 1 = 0$

  $x = 1$

  y-coordinate

  First, we MUST create a **new function** by reducing the original function. That is,

  $g(x) = x + 1$ with domain $(-\infty, \infty)$

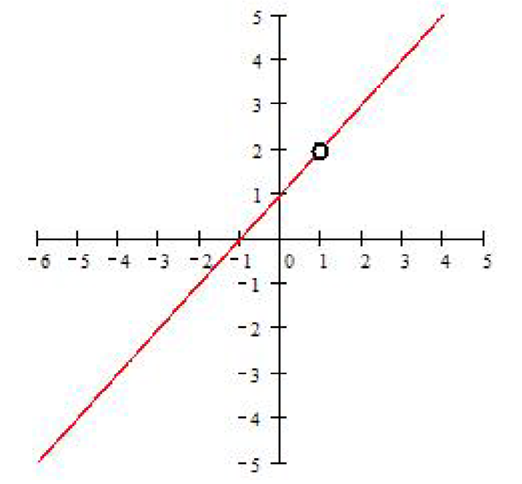
To find the y-coordinate of the hole, we simply replace ***x*** in the reduced function ***f*** with the x-coordinate of the hole

g(1) = 1 + 1 = 2

Finally, the coordinates of the hole are ***(1, 2)***.

Below is the graph of the function.

NOTE: The graphs of the functions ***h*** and ***g*** are almost identical. Both are linear. The only difference between them is that the function ***h*** has a **hole** at ***(1, 2)***.

	There is NO *horizontal* or *vertical asymptote*. Please observe the hole in the graph at ***(1, 2)***.