DETAILED SOLUTIONS AND CONCEPTS - LOGARITHM RULES AND BASIC PROPERTIES

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PLEASE NOTE THAT YOU CANNOT ALWAYS USE A CALCULATOR ON THE ACCUPLACER - COLLEGE-LEVEL MATHEMATICS TEST! YOU MUST BE ABLE TO DO SOME PROBLEMS WITHOUT A CALCULATOR!

Product Rule: $\log_b(MN) = \log_b M + \log_b N$

M and N can be any positive algebraic expression!

Example 1: $\log(3x) = \log 3 + \log x$

NOTE: $\log_b(3x) = \log_b 3x$ We usually do not place parentheses around a simple product. However, if you want to, you can always use parentheses.

Example 2: $\log[x(x-1)] = \log x + \log(x-1)$

WARNING!

$x\log_b 3 \neq \log_b 3x$

$\log_b(x+4) \neq \log_b x + \log_b 4$

$\log_b(x+4) \neq \log_b x + 4$

Quotient Rule: $\log_b \frac{M}{N} = \log_b M - \log_b N$

M and N can be any positive algebraic expression!

Example 1: $\log \frac{3}{x} = \log 3 - \log x$

Example 2: $\log \frac{x}{x-1} = \log x - \log(x-1)$

WARNING!

$$\log_b \frac{x}{4} \neq \frac{\log_b x}{\log_b 4}$$

$$\frac{\log_b x}{\log_b 4} \neq \log_b x - \log_b 4$$

$$\log_b (x-4) \neq \log_b x - \log_b 4$$

$$\log_b (x-4) \neq \log_b x - 4$$

Power Rule: $\log_b M^p = p \cdot \log_b M$

M can be any positive algebraic expression while ***p*** can be any expression, positive or negative.

NOTE: There is a multiplication sign between ***p*** and the logarithmic expression!!!

Example 1: $\log_7 x^2 = 2\log_7 x$

Example 2: $\log 3^{x-2} = (x-2)\log 3$

Example 3: $\ln 7^x = x\ln 7$

Example 4: $\log 3x^2 = \log 3 + 2\log x$

WARNING!

$$\log_b 3x^2 \neq 2\log_b 3x$$

$$\log_b 3^{x-2} \neq x - 2\log_b 3$$

Basic Logarithmic Properties

1. $\log_b b = 1$

   Example: $\log_7 7 = 1$ or $\log_{1000} 1000 = 1$ etc.

2. $\log_b 1 = 0$

   Example: $\log_7 1 = 0$ or $\log_{1000} 1 = 0$ etc.

3. $\log_b b^x = x$

Example: $\log_4 4^3 = 3$ or $\log_e \sqrt{e} = \log_e e^{1/2} = \frac{1}{2}$ or $\log_8 8 = 1$ etc.

4. $\log_b 0$ is undefined

Why? Let's assume that $\log_b 0$ equals some real number a. Changing to exponential form we would then get $0 = b^a$. However, given the positive real numbers b not equal to 1, no value for a would produce 0!

Point of Interest!

Proof of the Logarithm Rules

Let $\log_b M = m$ and $\log_b N = n$, then $M = b^m$ and $N = b^n$

Therefore, we can write

1. $\log_b MN = \log_b (b^m b^n) = \log_b b^{m+n} = m + n = \log_b M + \log_b N$

2. $\log_b \frac{M}{N} = \log_b \left(\frac{b^m}{b^n}\right) = \log_b b^{m-n} = m - n = \log_b M - \log_b N$

3. $\log_b M^p = \log_b (b^m)^p = \log_b b^{pm} = pm = p \log_b M$

Problem 1:

Write $\log[(x-2)(x+5)]$ in terms of simpler logarithms. Use the logarithm rules until no more can be applied.

Product Rule: $\log(x-2) + \log(x+5)$

Problem 2:

Write $\ln \frac{(6x+1)^7}{(4z+8)^6}$ in terms of simpler logarithms. Use the logarithm rules until no more can be applied.

Please note that lower case "ln" is "el n". Most font types make the lower case l ("el") look like an upper case I .

Quotient Rule: $ln(6x+1)^7 - ln(4z+8)^6$

Power Rule: $7\,ln(6x+1) - 6\,ln(4z+8)$ You have to keep the parentheses!

Problem 3:

Write $log_5 \frac{x-4}{x+6}$ in terms of simpler logarithms. Use the logarithm rules until no more can be applied.

Quotient Rule: $log_5(x-4) - log_5(x+6)$ You have to use the parentheses!

Problem 4:

Write $log\sqrt[5]{x}$ in terms of simpler logarithms. Use the logarithm properties until no more can be applied.

Let's first change the radical to exponential form

$log\, x^{1/5}$

Now we can apply the Power Rule to get $\frac{1}{5}\,log\, x$.

Problem 5:

Write $ln\sqrt[3]{\frac{x^2}{y^3 z}}$ in terms of simpler logarithms. Use the logarithm properties until no more can be applied.

First of all, let's write the cube root as the power of $\frac{1}{3}$.

$$ln\left(\frac{x^2}{y^3 z}\right)^{1/3}$$

Now, we need to apply the *Power Rule*.

$$\frac{1}{3}\,ln\left(\frac{x^2}{y^3 z}\right)$$

Next, we'll use the *Quotient Rule*! Be sure to notice that the power applies to all of the terms!!!

$$\frac{1}{3}(ln\, x^2 - ln\, y^3 z)$$

The second term in the parentheses is a product, therefore, we can now apply the *Product Rule*. Be sure to place grouping symbols appropriately because the minus applies to both ***ln y*** ^{3} and ***ln z***.

$$\tfrac{1}{3}[\ \ln x^2 - (\ln y^3 + \ln z)]$$

Lastly, we'll apply the *Power Rule* again.

$$\tfrac{1}{3}[\ 2\ln x - (3\ln y + \ln z)]$$

If you want you can distribute the minus sign as follows

$$\tfrac{1}{3}(2\ln x - 3\ln y - \ln z)$$

and, again if you want, you can distribute the $\frac{1}{3}$ to all terms

$$\tfrac{2}{3}\ln x - \ln y - \tfrac{1}{3}\ln z$$

Problem 6:

Let's write $\ln\sqrt[3]{\dfrac{x^2}{y^3 z}}$ in terms of simpler logarithms again. However, this time we'll use a different approach.

We do need to write the cube root as the power of $\frac{1}{3}$ first.

$$\ln\left(\frac{x^2}{y^3 z}\right)^{1/3}$$

However, next we'll distribute the $\frac{1}{3}$ to both the numerator and denominator.

$$\ln\frac{(x^2)^{1/3}}{(y^3 z)^{1/3}} = \ln\frac{x^{2/3}}{y\,z^{1/3}}$$

Now, we'll use the *Quotient Rule*.

$$\ln x^{2/3} - \ln yz^{1/3}$$

The second term is a product, therefore, we can now apply the *Product Rule*. Be sure to place grouping symbols appropriately because the minus applies to both ***ln y*** ^{3} and ***ln z***.

$$\ln x^{2/3} - (\ln y + \ln z^{1/3})$$

Lastly, we'll apply the *Power Rule* again.

$\frac{2}{3}\ln x - (\ln y + \frac{1}{3}\ln z)$

If you want you can distribute the minus as follows

$\frac{2}{3}\ln x - \ln y - \frac{1}{3}\ln z$

Problem 7:

Using ALL possible logarithm rules above, combine the following logarithmic expressions to one single expression.

$\log_3 2x + \log_3(x+1)$.

Using the Power Rule, we get $\log_3[2x(x+1)]$.

Problem 8:

Using ALL possible logarithm rules above, combine the following logarithmic expressions to one single expression

$\log_5 r + \log_5 s - \log_5 w$.

You must use the *Order of Operation*, that is perform addition and subtraction from left to right..

First, using the Product Rule, we get

$\log_5 rs - \log_5 w$

and finally, using the Quotient Rule, we find

$\log_5 \frac{rs}{w}$

Problem 9:

Using ALL possible logarithm rules above, combine the following logarithmic expressions to one single expression

$\frac{1}{3}\ln y - 3\ln 2 + 8\ln z$.

The Product and the Quotient Rule can only be used if the coefficient of the logarithmic term is equal to *1*. Therefore, we must use the Power Rule first.

$\ln y^{1/3} - \ln 2^3 + \ln z^8$ or $\ln y^{1/3} - \ln 8 + \ln z^8$

Now, using the Quotient Rule, we find

$$\ln\frac{y^{1/3}}{8}+\ln z^{8}$$

and finally, using the Product Rule, we get

$$\ln\frac{y^{1/3}z^{8}}{8}$$

Problem 10:

Using ALL possible logarithm rules above, combine the following logarithmic expressions to one single expression

$$5\ln w-4\ln x-\tfrac{1}{2}\ln y$$

$\ln w^{5}-\ln x^{4}-\ln\sqrt{y}$. Remember $y^{1/2}=\sqrt{y}$!!!

$$\ln\frac{w^{5}}{x^{4}}-\ln\sqrt{y}$$

Applying the Quotient Rule again, we get

$$\ln\frac{\frac{w^{5}}{x^{4}}}{\sqrt{y}}$$

Since any compound fraction must be reduced to simple form, we find

$$\ln\frac{w^{5}}{x^{4}\sqrt{y}}$$

Remember, $\dfrac{\frac{w^{5}}{x^{4}}}{\sqrt{y}}$ means $\dfrac{w^{5}}{x^{4}}\div\dfrac{\sqrt{y}}{1}$ and dividing by a number is the same as multiplying by its reciprocal, so that $\dfrac{w^{5}}{x^{4}}\cdot\dfrac{1}{\sqrt{y}}=\dfrac{w^{5}}{x^{4}\sqrt{y}}$!!!

Problem 11:

Using ALL possible logarithm rules above, combine the following logarithmic expressions to one single expression.

$$\tfrac{1}{2}\log(x-3) - 3\log(x^2+2) - \tfrac{1}{3}\log(x+1)$$

The Product and the Quotient Rule can only be used if the coefficient of the logarithmic term is equal to *1*. Therefore, we must use the Power Rule first.

$$\log(x-3)^{1/2} - \log(x^2+2)^3 - \log(x+1)^{1/3}$$

You must use the *Order of Operation*, that is perform addition and subtraction from left to right.

$$\log\left[\frac{(x-3)^{1/2}}{(x^2+2)^3}\right] - \log(x+1)^{1/3}$$

Now we will use the Quotient Rule again, to find

$$\log\left[\frac{\frac{(x-3)^{1/2}}{(x^2+2)^3}}{(x+1)^{1/3}}\right]$$

However, any compound fraction must be simplified. Since dividing by a number is equivalent to multiplying by its reciprocal, we get

$$\log\left[\frac{(x-3)^{1/2}}{(x^2+2)^3(x+1)^{1/3}}\right]$$

Problem 12:

Evaluate the following common and natural logarithms without a calculator. Instead, use one of the four basic logarithm properties stated above.

a. $\log 100$

Property 3: $\log 100 = \log 10^2 = 2$

b. $\log\sqrt{10}$

Property 3: $\log\sqrt{10} = \log 10^{1/2} = \frac{1}{2}$

c. $\log 1$

Property 2: $\log 1 = 0$

d. $\ln e^{0.63}$

Property 3: $\ln e^{0.63} = 0.63$

e. $\ln e$

Property 1: $\ln e = \log_e e^1 = 1$

f. $\ln 1$

Property 2: $\ln 1 = 0$

g. $\log_2 \frac{1}{2}$

Remember that $\frac{1}{2} = 2^{-1}$, so that we can write $\log_2 2^{-1} = -1$ using Property 3 above.

h. $\log 0$

Property 4: The logarithm of any number is undefined.