**DETAILED SOLUTIONS AND CONCEPTS - AN INTRODUCTION TO CONIC SECTIONS**

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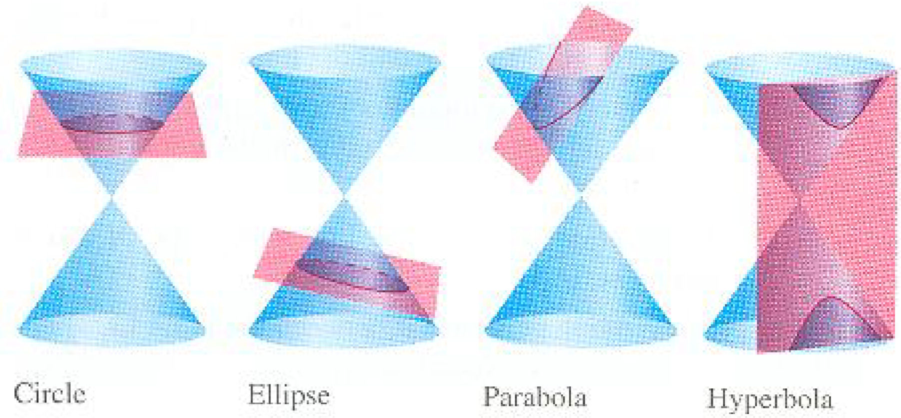
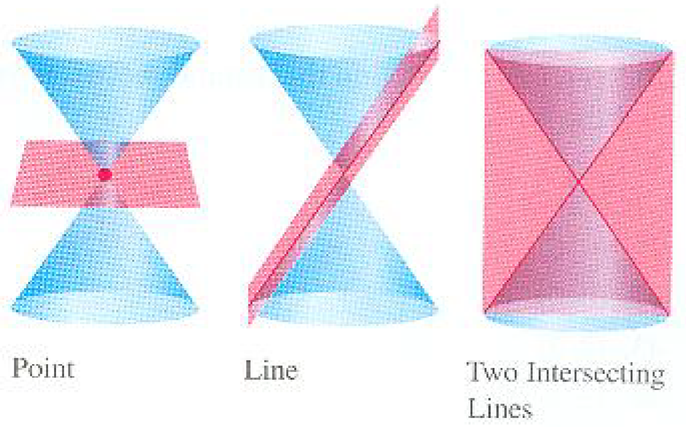
**Please Send Questions and Comments to ingrid.stewart@csn.edu. Thank you!**

**PLEASE NOTE THAT YOU CANNOT ALWAYS USE A CALCULATOR ON THE ACCUPLACER - COLLEGE-LEVEL MATHEMATICS TEST! YOU MUST BE ABLE TO DO SOME PROBLEMS WITHOUT A CALCULATOR!**

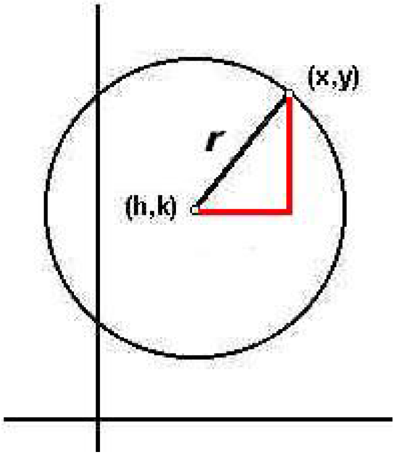
A conic section (or simply a conic) can be described as the intersection of a plane with two cones stacked tip to tip (a double-napped cone). The point where the two tips meet is called the **vertex**. The most common intersections are the circle, the ellipse, the parabola, and the hyperbola. In these cases, the plane does **NOT** pass through the vertex.




When the plane passes through the vertex, the resulting figure is a *degenerate conic* as shown below:



**CIRCLES**




**Standard Form of the Equation:**

This equation is derived by using the Pythagorean Theorem on the right triangle as shown in the picture.

$$(x-h)^2 + (y-k)^2 = r^2$$

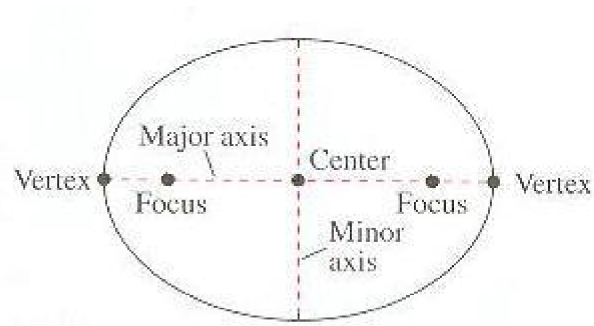
Coordinates of the Vertex: $(h, k)$

Radius: $r$

**ELLIPSES**

Ellipses have many practical and aesthetic uses. For example, machine gears, supporting arches, and acoustical designs often involve elliptical shapes. The orbits of satellites and planets are also ellipses.

**Ellipses with a horizontal major axis:**




**Standard Equation of the Ellipse:**

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, where $0 < b < a$ and **major axis is horizontal (longer axis)**

**Note that when the larger value is the denominator of the x-variables, the major axis is horizontal!**

The two foci lie on the major axis, each $c$ units from the center, where $c^2 = a^2 - b^2$. The two vertices also lie on the major axis each $a$ units from the center. That is, the center of the ellipse lies exactly in between the two foci and the two vertices.

Coordinates of the Center: ***(h, k)***

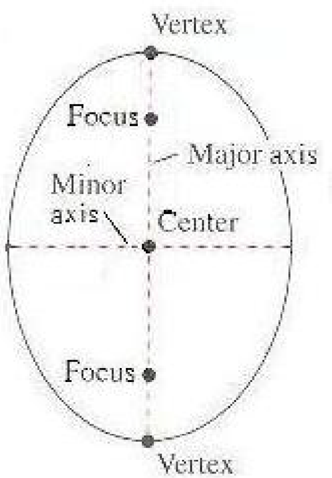
Coordinates of the Vertices: ***(h + a, k)*** and ***(h - a, k)***

Coordinates of the Foci: ***(h + c, k)*** and ***(h - c, k)***

Length of the major axis is ***2a***

Length of the minor axis is ***2b***

**Ellipses with a vertical major axis:**




**Standard Equation of the Ellipse:**

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$, where $0 < b < a$ and **major axis is vertical (longer axis)**

**Note that when the larger value is the denominator of the y-variables, the major axis is vertical!**

The two foci lie on the major axis, each $c$ units from the center, where $c^2 = a^2 - b^2$. The two vertices also lie on the major axis each $a$ units from the center. That is, the center of the ellipse lies exactly in between the two foci and the two vertices.

Coordinates of the Center: ***(h, k)***

Coordinates of the Vertices: ***(h, k + a)*** and ***(h, k - a)***

Coordinates of the Foci: ***(h, k + c)*** and ***(h, k - c)***, where $c^2 = a^2 - b^2$

Length of the major axis is ***2a***

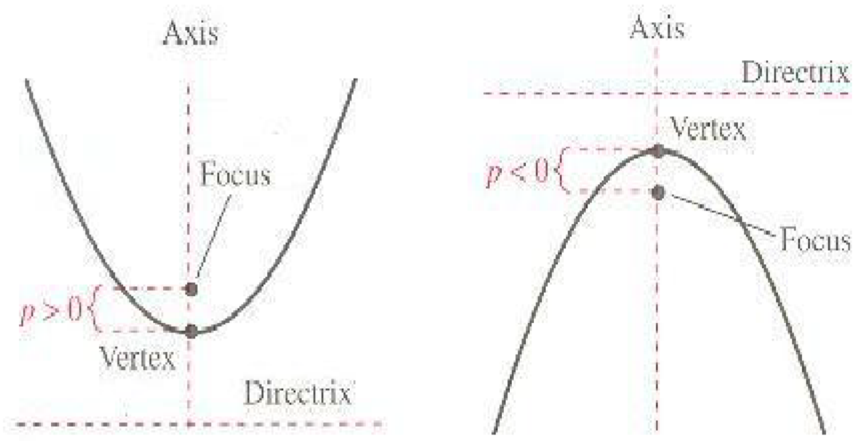
Length of the minor axis is ***2b***

**PARABOLAS**

Parabolas occur in a wide variety of applications. For example, a parabolic reflector can be formed by revolving a parabola around its axis. The resulting surface has the property that all incoming rays parallel to the axis are reflected through the focus of the parabola. This is the principle behind the construction of the parabolic mirrors used in reflecting telescopes.

Tangent lines to parabolas have special properties related to the use of parabolas in constructing reflective surfaces. A line is tangent to a parabola at a point on the parabola if the line touches, but does not cross, the parabola at the point.

**Parabolas open up and open down:**




**Standard Form of the Equation:**

$(x - h)^2 = 4p(y - k)$, where ***p*** can be negative (open downward) or positive (open upward), but is never equal to ***0***.

**NOTE: In conics, the number *4* is always part of the coefficient *a* of the standard equation of the parabola.**

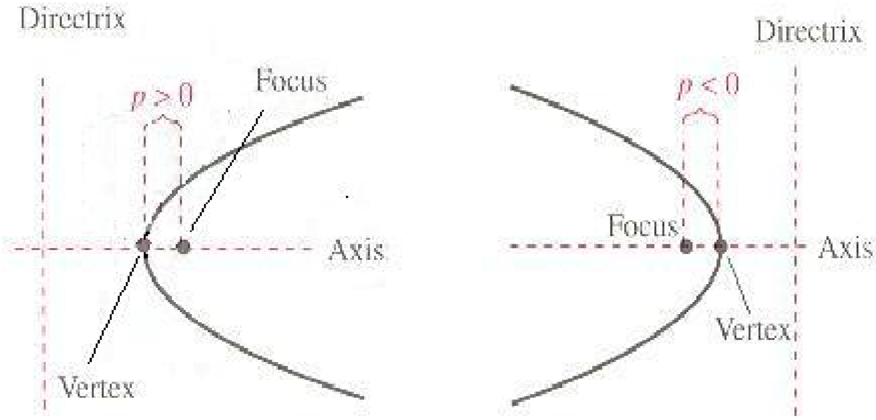
Coordinates of the Vertex: ***(h, k)***

Coordinates of the Focus: ***(h, k + p)***

Equation of the Directrix: ***y = k - p***

Equation of the Axis: ***x = h***

**Parabolas open to the right and open to the left:**




**Standard Form of the Equation:**

$(y - k)^2 = 4p(x - h)$, where $p$ can be negative (open left) or positive (open right), but is never equal to $0$.

Coordinates of the Vertex: $(h, k)$

Coordinates of the Focus: $(h + p, k)$

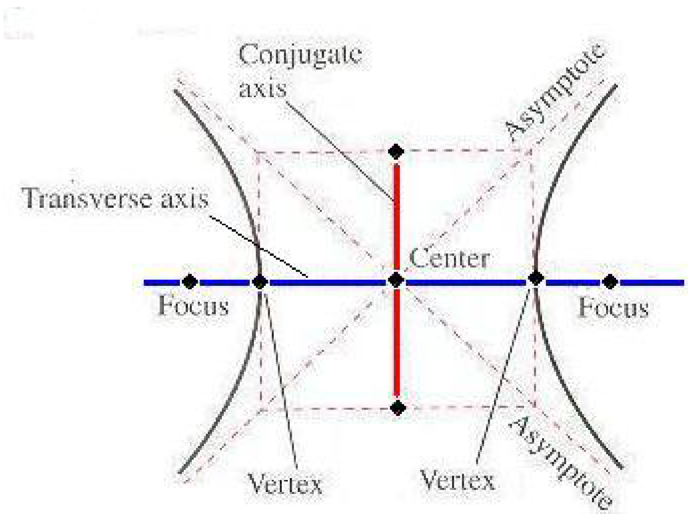
Equation of the Directrix: $x = h - p$

Equation of the Axis: $y = k$

**HYPERBOLAS**

The properties of hyperbolas can be used in radar and other detection systems. Another interesting application of conic sections involves the orbits of comets in our solar system. Of the 610 comets identified prior to 1970, 245 have elliptical orbits, 295 have parabolic orbits, and 70 have hyperbolic orbits. Comets with elliptical orbits, such as Halley's comet, are the only ones that remain in our solar system.

**Hyperbolas with a horizontal transverse axis:**




**Standard Equation of the Hyperbola:**

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

, where $a = b$ or $a \neq b$ and **transverse axis is horizontal**

**Note that the negative sign in front of the y-variables determines that the transverse axis is horizontal!**

The two foci lie on the transverse axis, each $c$ units from the center, where $c^2 = a^2 + b^2$. The two vertices also lie on the transverse axis each $a$ units from the center. That is, the center of the hyperbola lies exactly in between the two foci and the two vertices.

Coordinates of the Center: $(h, k)$

Coordinates of the Vertices: $(h + a, k)$ and $(h - a, k)$

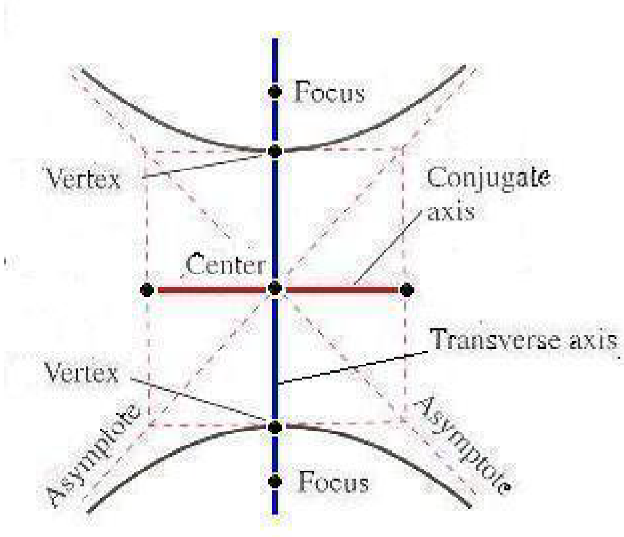
Coordinates of the Foci: $(h + c, k)$ and $(h - c, k)$

Length of the transverse axis is $2a$

Length of the conjugate axis is $2b$

Equation of the Asymptotes: $y = k \pm \frac{b}{a}(x-h)$

**Hyperbolas with a vertical transverse axis:**




**Standard Equation of the Hyperbola:**

$\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$, where ***a = b*** or ***a ≠ b*** and **transverse axis is vertical**

**Note that the negative sign in front of the x-variables determines that the transverse axis is vertical!**

The two foci lie on the transverse axis, each ***c*** units from the center, where $c^2 = a^2 + b^2$. The two vertices also lie on the transverse axis each ***a*** units from the center. That is, the center of the hyperbola lies exactly in between the two foci and the two vertices.

Coordinates of the Center: ***(h, k)***

Coordinates of the Vertices: ***(h, k + a)*** and ***(h, k - a)***

Coordinates of the Foci: ***(h, k + c)*** and ***(h, k - c)***, where $c^2 = a^2 + b^2$

Length of the transverse axis is ***2a***

Length of the conjugate axis is ***2b***

Equation of the Asymptotes: $y = k \pm \dfrac{a}{b}(x-h)$

## IDENTIFYING A CONIC SECTION WRITTEN IN GENERAL FORM

Excluding the degenerate cases as well as rotations of the axes, let

$$Ax^2 + Bx + Cy^2 + Dy + E = 0$$

be the general form of a conic section, where $A$ and $C$ cannot both be equal to $0$.

Then this equation

a. defines a parabola if $AC = 0$

b. defines a hyperbola if $AC < 0$

c. defines an ellipse (or circle) if $AC > 0$

The difference between an ellipse and a circle:

- In a circle, the coefficients of $x^2$ and $y^2$ are equal when on the same side.
- In an ellipse, the coefficients of $x^2$ and $y^2$ are **NOT** equal when on the same side.

**The following are equations of conic sections in general form. Identify them by name (circle, ellipse, parabola, hyperbola).**

**Problem 1:**

$$x^2 + y^2 = 9$$

$AC = 1 > 0$ - circle because the coefficients of $x^2$ and $y^2$ are equal when on the same side

**Problem 2:**

$$4x^2 + 9y^2 = 36$$

$AC = 36 > 0$ - ellipse because the coefficients of $x^2$ and $y^2$ are NOT equal when on the same side

**Problem 3:**

$$-4x^2 + 9y^2 = 36$$

$AC = -36 < 0$ - hyperbola

**Problem 4:**

$$y = \frac{1}{8}x^2$$

***AC = 0*** - parabola (C = 0) !!!

**Problem 5:**

$$x = \frac{1}{8}y^2$$

***AC = 0*** - parabola (A = 0) !!!

**Problem 6:**

$$9x^2 + 4y^2 = 36$$

***AC = 36 > 0*** - ellipse because the coefficients of $x^2$ and $y^2$ are NOT equal when on the same side

**Problem 7:**

$$y = 4x^2 - 3x - 16$$

***AC = 0*** - parabola (C = 0) !!!

**Problem 8:**

$$x^2 + y^2 - 2x - 7 = 0$$

***AC = 1 > 0*** - circle because the coefficients of $x^2$ and $y^2$ are equal when on the same side

**Problem 9:**

$$4x^2 - 6y^2 + 7y = 9$$

***AC = -24 < 0*** - hyperbola

**Problem 10:**

$$4x^2 + 6y^2 - 3x + 7y = 16$$

***AC = 24 > 0*** - ellipse because the coefficients of $x^2$ and $y^2$ are NOT equal when on the same side

**Problem 11:**

$$x = -\tfrac{1}{8}y^2$$

***AC = 0*** - parabola (A = 0) !!!

**Problem 12:**

$$4x^2 - 9y^2 = 36$$

***AC = -36 < 0*** - hyperbola

**Problem 13:**

$$x = 6y^2 + 7y + 5$$

***AC = 0*** - parabola (A = 0) !!!

**Problem 14:**

$$-4x^2 + y^2 - 3x + 7y = 3$$

***AC = -4 < 0*** - hyperbola

**Problem 15:**

$$y = -\tfrac{1}{8}x^2$$

***AC = 0*** - parabola (C = 0) !!!