$$\lim_{x \to \infty} \int_2^3 \frac{1}{dx} dy$$

THE GENERAL POWER RULE AND THE LOGARITHMIC RULE FOR INTEGRATION

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The General Power Rule for Integration

$$\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C,$$

for all real numbers n except $n \neq -1$

NOTE: This formula undoes the *General Power Rule* used in differentiation!

This rule is usually restated with a substitution as follows:

Let $u = f(x)$ be a function of x. This implies that $\frac{du}{dx} = f'(x)$ and $du = f'(x)\,dx$.

Then $\int u\ du = \frac{u^{n+1}}{n+1} + C$, for all real numbers n except $n \neq -1$

The Log Rule for Integration

$$\int [f(x)]^{-1} f'(x)\,dx = \ln|f(x)| + C$$ for $n = -1$

This rule is also usually restated with a substitution as follows:

Let $u = f(x)$ be a function of x. This implies that $\frac{du}{dx} = f'(x)$ and $du = f'(x)\,dx$.

Then $\int u^{-1} du = \ln|u| + C$

WHY IS SUBSTITUTION NECESSARY?

Substitution allows us to decide whether or not we can use the *General Power Rule or the Log Rule for Integration*. Since the *f ' (x)* factor in the integrand is not always readily apparent, the *du*-equation will immediately point out the *f ' (x)* factor to you. If they match, except maybe for a difference in constant multiple, most likely you can use the *General Power Rule or the Log Rule for Integration*.

Guidelines for Using Substitution

1. Write your integrand as a product, if necessary.

2. Decide which of the factors of the integrand to substitute with ***u***.

   NOTE: This is your decision. As a beginner you might have to make more than one attempt to pick the "correct" factor that would facilitate the *General Power Rule or the Log Rule of Integration*.

3. Compute ***du***.

   a. If the right side of the *du*-equation matches the remaining factor(s) in the integrand (the ones not used for ***u***), except maybe for a difference in constant factor, rewrite the integral as

   $$\int u^n du$$

   and find the family of antiderivatives using

   $$\int u^n du = \frac{u^{n+1}}{n+1} + C.$$

   b. If the right side of the *du*-equation DOES NOT match the remaining factor, substitute a different factor with ***u*** and compute ***du*** again. If there is still no match, you CANNOT use the *General Power Rule or the Log Rule for Integration*. Other integration procedures will have to be used.

4. If 3.a. applies, change the antiderivative back to the original variables.

NOTE: Before using *u*-substitution you should ALWAYS try to use the basic integration formulas discussed previously.

Problem 1:

Integrate $\int 4x(2x^2+3)^{50}\ dx$. Note that "integrate" actually means to find the antiderivative for the function $f(x) = 4x(2x^2+3)^{50}$!!!

Reasons for trying to use the *General Power Rule*:

1. The integrand contains an algebraic expression raised to a power.

2. Because this algebraic expression is "more" than just a single variable with coefficient 1, the integrand might be a derivative found through the use of the *General Power Rule for Differentiation*.

Recall:

Let ***u*** be a function of ***x***. If $f(x) = u^n$, then $f'(x) = u' \cdot n u^{n-1}$.

Before we can proceed with the *General Power Rule for Integration,* we must check to see if ***u'*** is the ONLY other factor of the integrand. This is done with the help of *u*-substitution.

$u = 2x^2 + 3$

$du/dx = 4x$

$du = 4x\ dx$

Since the right side of the *du*-equation matches the remaining factors ***4***, ***x***, and ***dx*** of the integrand exactly, we are now assured that we can use the *General Power Rule for Integration* to find the family of antiderivatives.

We proceed to write the integral in terms of ***u*** and evaluate as follows:

$\int u^{50}\ du = 1/51\ u^{51} + C$

and since the variable of integration is ***x***, we'll convert back to the original variable to find that

$$\int 4x(2x^2+3)^{50}\ dx = \frac{1}{51}(2x^2+3)^{51} + C$$

Therefore, the antiderivative is $$F(x) = \frac{1}{51}(2x^2+3)^{51} + C$$

How come there is no factor 4x in the antiderivative and where did the factor $1/_{51}$ come from?

To explain this, let's find the derivative of

$$F(x) = \tfrac{1}{51}(2x^2 + 3)^{51} + C.$$

Using the *General Power Rule for Differentiation*, we let

$u = 2x^2 + 3$, then ***$u' = 4x$***.

therefore

$$F'(x) = \tfrac{1}{51}\{4x[51(2x^2+3)^{50}]\} = 4x(2x^2+3)^{50} \text{ !!!}$$

Here we see that the factor 4x "appears" in the derivative and the factor $1/_{51}$ "vanishes" due to the make-up of the General Power Rule for Differentiation. Therefore, when undoing differentiation with the Chain Rule for Integration the factor 4x must "vanish" and the factor $1/_{51}$ must "appear" again !!!!!!

Problem 2:

Evaluate $\int 8x(2x^2+3)^{50}\, dx$. Note that "evaluate" actually means to find the antiderivative for the function $f(x) = 8x(2x^2+3)^{50}$!!!

This integrand is very similar to the one in Problem 1. The only difference is that it contains a constant factor ***8*** instead of a factor ***4***.

However, we can still attempt to use the *General Power Rule for Integration* to evaluate the integral!!!

Using *u*-substitution, we find

$u = 2x^2 + 3$

$du = 4x\, dx$

The right side of the *du*-equation does not quite match the remaining two factors of the integrand because the constant factor of the integrand is ***8***. This is easily fixed by multiplying the *du*-equation as follows:

$2du = 8x\, dx$

NOTE: You are allowed to multiply du by a constant. HOWEVER, YOU ARE NOT ALLOWED TO MULTIPLY du BY A VARIABLE !!!

Since the right side of the *du*-equation matches the remaining factors ***8***, ***x***, and ***dx*** of the integrand exactly, we are now assured that we can use the *General Power Rule for Integration* to find the family of antiderivatives.

We proceed to write the integral in terms of ***u*** and evaluate as follows:

$2\int u^{50}\, du = 2/51\, u^{51} + C$

and since the variable of integration is $\boldsymbol{x}$, we'll convert back to the original variable to find that

$$\int 8x(2x^2+3)^{50}\,dx = \tfrac{2}{51}(2x^2+3)^{51} + C$$

Therefore, the antiderivative is $$F(x) = \frac{2}{51}(2x^2+3)^{51} + C$$

Problem 3:

Evaluate $$\int x(2x^2+3)^{50}\,dx$$

This integrand is very similar to the one in Problem 1. The only difference is that it contains a constant factor of $\boldsymbol{1}$ instead of a factor of $\boldsymbol{4}$.

However, we can still attempt to use the *General Power Rule for Integration* to evaluate the integral. Using *u*-substitution, we find

$$u = 2x^2 + 3$$

$$du = 4x\,dx$$

The right side of the *du*-equation contains the remaining factors $\boldsymbol{x}$ and $\boldsymbol{dx}$ of the integrand, but the integrand does not have a constant factor. This is easily fixed by multiplying the *du*-equation as follows:

$$du = x\,dx$$

Since the right side of the *du*-equation matches the remaining factors $\boldsymbol{x}$ and $\boldsymbol{dx}$ of the integrand exactly, we are now assured that we can use the *General Power Rule for Integration* to find the family of antiderivatives.

We proceed to write the integral in terms of $\boldsymbol{u}$ and evaluate as follows:

$$\int u^{50}\,du = (1/51\, u^{51}) + C$$

and since the variable of integration is $\boldsymbol{x}$, we'll convert back to the original variable to find that

$$\int x(2x^2+3)^{50}\,dx = \tfrac{1}{204}(2x^2+3)^{51} + C$$

Therefore, the antiderivative is $$F(x) = \frac{1}{204}(2x^2+3)^{51} + C$$

Problem 4:

Evaluate $\int (2x^2 + 3)^{50}\ dx$

This integrand also SEEMS very similar to the one in Problem 1. The only difference is that it does not contain a constant factor and a variable.

However, we can still attempt to use the *General Power Rule for Integration* to evaluate the integral. Using *u*-substitution, we find

$$u = 2x^2 + 3$$

$$du = 4x\ dx$$

The right side of the *du*-equation does not match the remaining factors of the integrand. First of all the constant factor of the integrand is ***1***. Secondly, the integrand does NOT have a factor ***x***, but the *du*-equation does. Taking care of the constant factor isn't a problem as we have seen in earlier examples. However, there is NOTHING that can be done to take the factor ***x*** out of the *du*-equation !!!

NO, you cannot divide both sides of the *du*-equation by *x* !!! That would not be a mathematically-sound procedure!

In this case, we have to give up and hopefully find a different way to evaluate this integral! More advanced methods of integration will be discussed in Calculus II. However, realize, that sometimes an antiderivative simply cannot be found!

Problem 5:

Evaluate $\int (2x + 3)^2\ dx$

This integrand is similar to the one in Problem 4. The only difference is that the ***x*** is raised to the first power and not to the second power!

Let's attempt to use the *General Power Rule for Integration* to evaluate the integral. Using *u*-substitution, we find

$$u = 2x + 3$$

$$du = 2\ dx$$

The right side of the *du*-equation contains the remaining factor ***dx*** of the integrand, but the integrand does not have a constant factor. This is easily fixed by multiplying the *du*-equation as follows:

$$du = dx$$

Since the right side of the *du*-equation matches the remaining factor ***dx*** of the integrand exactly, we are now assured that we can use the *General Power Rule for Integration* to find the family of antiderivatives.

We proceed to write the integral in terms of ***u*** and evaluate as follows:

$$\int u^2\, du = \left(\tfrac{1}{3} u^3\right) + C$$

and since the variable of integration is ***x***, we'll convert back to the original variable to find that

$$\int (2x+3)^2\, dx = \tfrac{1}{6}(2x+3)^3 + C = \tfrac{4}{3}x^3 + 6x^2 + 9x + \tfrac{9}{2} + C$$

Therefore, the antiderivative is $F(x) = \dfrac{4}{3}x^3 + 6x^2 + 9x + \dfrac{9}{2} + C$

Please note that the power is small enough to where we also could have used FOIL to expand the product. This, of course, would not work given a power of 50, for example!

$$\int (2x+3)^2\, dx = \int (4x^2 + 12x + 9)\, dx$$
$$= \tfrac{4}{3}x^3 + 6x^2 + 9x + C$$

Therefore, the antiderivative is $F(x) = \dfrac{4}{3}x^3 + 6x^2 + 9x + C$

Comparing the antiderivatives derived from the two different methods, we find that one contains the constant $\tfrac{9}{2}$, whereas the other one does not.

Are the two antiderivatives equivalent? Yes they are! One antiderivative simply shows more of the constant factor ***C*** than the other one. As a matter of fact we could eliminate the constant $\tfrac{9}{2}$.

Problem 6:

Evaluate $\displaystyle\int \frac{t^2 + t}{(2t^3 + 3t^2)^4}\, dt$

First let's first write the integrand as a product

$$\int \frac{t^2 + t}{(2t^3 + 3t^2)^4}\, dt = \int (t^2 + t)(2t^3 + 3t^2)^{-4}\, dt$$

Reasons for trying to use the *General Power Rule*:

1. The integrand contains an algebraic expression raised to a power.

2. Because this algebraic expression is "more" than just a single variable with coefficient 1, the integrand might be a derivative found through the use of the *General Power Rule for Differentiation*.

Here the choice for u might not be as apparent as in the previous examples. A good rule of thumb, although no guarantee, is to use the quantity to the higher **absolute** power. If your choice does not seem to work out, you might try the other quantity!

Using *u*-substitution as follows, we find

$$u = 2t^3 + 3t^2$$

$$du = (6t^2 + 6t)\,dt = 6(t^2 + t)\,dt$$

The right side of the *du*-equation does not quite match the remaining factors of the integrand because the constant factor of the integrand is ***1***. This is easily fixed by multiplying the *du*-equation as follows:

$$\tfrac{1}{6}\,du = (t^2 + t)\,dt$$

Since now the right side of the *du*-equation matches the remaining factors dt and $(t^2 + t)$ of the integrand exactly, we are now assured that we can use the *General Power Rule for Integration* to find the family of antiderivatives.

We proceed to write the integral in terms of u and evaluate as follows:

$$\tfrac{1}{6}\int u^{-4}\,du = \tfrac{1}{6}\left(-\tfrac{1}{3}u^{-3}\right) + C$$

and since the variable of integration is t, we'll convert back to the original variable to find that

$$\int \frac{t^2 + t}{(2t^3 + 3t^2)^4}\,dt = \int (t^2 + t)(2t^3 + 3t^2)^{-4}\,dt = -\tfrac{1}{18}(2t^3 + 3t^2)^{-3} + C$$

and $$\int \frac{t^2 + t}{(2t^3 + 3t^2)^4}\,dt = \frac{-1}{18(2t^3 + 3t^2)^3} + C$$

Therefore, the antiderivative is $$F(t) = \frac{-1}{18(2t^3 + 3t^2)^3} + C$$

Problem 7:

Evaluate $\int \frac{1}{x^2}\left(1+\frac{1}{x}\right)^3 dx$

First let's change the rational terms to exponential terms

$$\int \frac{1}{x^2}\left(1+\frac{1}{x}\right)^3 dx = \int x^{-2}(1+x^{-1})^3 dx$$

Reasons for trying to use the *General Power Rule*:

1. The integrand contains an algebraic expression raised to a power.

2. Because this algebraic expression is "more" than just a single variable with coefficient 1, the integrand might be a derivative found through the use of the *General Power Rule for Differentiation*.

Using *u*-substitution as follows, we find

$u = 1 + x^{-1}$

$du = -x^{-2}\, dx$

The right side of the *du*-equation contains the remaining factors x^{-2} and dx of the integrand, but the integrand does not have a constant factor of -1. This is easily fixed by multiplying the *du*-equation as follows:

$- du = x^{-2}\, dx$

Since now the right side of the *du*-equation matches the remaining factors of the integrand exactly, we are now assured that we can use the *General Power Rule for Integration* to find the family of antiderivatives.

We proceed to write the integral in terms of u and evaluate as follows:

$- \int u^3\, du = - (u^4) + C$

and since the variable of integration is x, we'll convert back to the original variable to find that

$$\int \frac{1}{x^2}\left(1+\frac{1}{x}\right)^{-3} dx = \int x^{-2}(1+x^{-1})^{-3} dx = -\tfrac{1}{4}(1+x^{-1})^4 + C$$

and $\int \frac{1}{x^2}\left(1+\frac{1}{x}\right)^3 dx = -\tfrac{1}{4}\left(1+\frac{1}{x}\right)^4 + C$

Therefore, the antiderivative is $$F(x) = -\frac{1}{4}\left(1 - \frac{1}{x}\right)^{4} + C$$

Problem 8:

Evaluate $$\int \frac{v}{\sqrt{9 - v^{2}}}\,dv$$

First let's rewrite the integrand as a product and using exponents

$$\int \frac{v}{\sqrt{9 - v^{2}}}\,dv = \int v(9 - v^{2})^{-1/2}\,dv$$

Reasons for trying to use the *General Power Rule*:

1. The integrand contains an algebraic expression raised to a power.

2. Because this algebraic expression is "more" than just a single variable with coefficient 1, the integrand might be a derivative found through the use of the *General Power Rule for Differentiation*.

Using *u*-substitution as follows, we find

$u = 9 - v^{2}$

$du = -2v\,dv$

The right side of the *du*-equation contains the remaining factors ***v*** and ***dv*** of the integrand, but the integrand does not have a constant factor of ***-2***. This is easily fixed by multiplying the *du*-equation as follows:

$-du = v\,dv$

Since now the right side of the *du*-equation matches the remaining factors of the integrand exactly, we are now assured that we can use the *General Power Rule for Integration* to find the family of antiderivatives.

We proceed to write the integral in terms of ***u*** and evaluate as follows:

$-\int u^{-}\,du = -(2u) + C$

and since the variable of integration is ***v***, we'll convert back to the original variable to find that

$$\int \frac{v}{\sqrt{9 - v^{2}}}\,dv = -\sqrt{9 - v^{2}} + C$$

$$F(v) = -\sqrt{9 - v^2} + C$$

Therefore, the antiderivative is

Problem 9:

Evaluate $\int \sin^3 x \cos x \, dx$

Reasons for trying to use the *General Power Rule*:

1. The integrand does not match any of the basic trigonometric integration formulas!

NOTE:

$\int \sec x \tan x \, dx$ is also a product, but it matches a basic trigonometric integration formula. In this case, we would NOT use the *General Power Rule for Integration*!

2. The integrand contains a trigonometric expression raised to a power.

Using *u*-substitution as follows, we find

$u = \sin x$

$du = \cos x \, dx$

Since the right side of the *du*-equation matches the remaining factors dx and $\cos x$ of the integrand exactly, we are now assured that we can use the *General Power Rule for Integration* to find the family of antiderivatives.

We proceed to write the integral in terms of u and evaluate as follows:

$\int u^3 \, du = u^4 + C$

and since the variable of integration is x, we'll convert back to the original variable to find that

$$\int \sin^3 x \cos x \, dx = \tfrac{1}{4} \sin^4 x + C$$

Therefore, the antiderivative is $F(x) = \dfrac{1}{4} \sin^4 x + C$

Problem 10:

Evaluate $\int \csc^2 x \cot x \, dx$

Reasons for trying to use the *General Power Rule*:

1. The integrand does not match any of the basic trigonometric integration formulas!

2. The integrand contains a trigonometric expression raised to a power.

Here you have to be very careful! While it was mentioned earlier that a good rule of thumb is to use the quantity to the higher **absolute** power, there is really no guarantee that this will help us find the antiderivative.

a. Let's investigate what will happen if we use the following substitution:

$$u = \csc x$$

$$du = -\csc x \cot x \, dx$$

It seems that the *du*-equation does not match the remaining factors dx and cot x of the integrand. However, watch what happens if we rewrite the integrand as follows

$$\int \csc^2 x \cot x \, dx = \int \csc x (\csc x \cot x) \, dx$$

Now the right side of the *du*-equation contains the remaining factors ***dx***, ***csc x***, and ***cot x*** of the integrand, but the integrand does not have a constant factor of ***-1***. This is easily fixed by multiplying the *du*-equation as follows:

$$-du = \csc x \cot x \, dx$$

Now, the right side of the *du*-equation matches the remaining factors of the integrand exactly. We are now assured that we can use the *General Power Rule for Integration* to find the family of antiderivatives.

We proceed to write the integral in terms of ***u*** and evaluate as follows:

$$-\int u \, du = -u^2 + C$$

and since the variable of integration is ***x***, we'll convert back to the original variable to find that

$$\int \csc^2 x \cot x \, dx = \int \csc x (\csc x \cot x) \, dx = -\tfrac{1}{2} \csc^2 x + C$$

Therefore, the antiderivative is $F(x) = -\dfrac{1}{2}\csc^2 x + C$

b. Now let's investigate what will happen if we use the substitution

$u = \cot x$

$du = -\csc^2 x\, dx$

The right side of the *du*-equation contains the remaining factors $\boldsymbol{\csc^2 x}$ and d$\boldsymbol{x}$ of the integrand, but the integrand does not have a constant factor of ***-1***. This is easily fixed by multiplying the *du*-equation as follows:

$-du = \csc^2 x\, dx$

Now, the right side of the *du*-equation matches the remaining factors of the integrand exactly. We are now assured that we can use the *General Power Rule for Integration* to find the family of antiderivatives.

We proceed to write the integral in terms of ***u*** and evaluate as follows:

$-\int u\, du = -u^2 + C$

and since the variable of integration is ***x***, we'll convert back to the original variable to find that

$$\int \csc^2 x \cot x\, dx = -\tfrac{1}{2}\cot^2 x + C$$

Therefore, the antiderivative is $F(x) = -\dfrac{1}{2}\cot^2 x + C$

When comparing the antiderivatives derived from the two different methods used in (a) and (b), they do not seem to be the same.

That is,

(a) $\int \csc^2 x \cot x\, dx = \int \csc x(\csc x \cot x)\, dx = -\tfrac{1}{2}\csc^2 x + C$

(b) $\int \csc^2 x \cot x\, dx = -\tfrac{1}{2}\cot^2 x + C$

However, we can make them be the same by using the *Pythagorean Identity* $\cot^2 x = \csc^2 x - 1$.

Then the antiderivative in (b) can be rewritten as

$$\int \csc^2 x \cot x \, dx = -\tfrac{1}{2}\cot^2 x + C$$
$$= -\tfrac{1}{2}(\csc^2 x - 1) + C$$
$$= -\tfrac{1}{2}\csc^2 x + \tfrac{1}{2} + C$$

Comparing the antiderivatives derived now, we find that one contains the constant $\frac{1}{2}$, whereas the other one does not.

Are the two antiderivatives equivalent? Yes they are! One antiderivative simply shows more of the constant factor ***C*** than the other one. As a matter of fact we could eliminate the constant $\frac{1}{2}$ in (b).

Problem 11:

Evaluate $\int \sin x \cos x \, dx$

Reasons for trying to use the *General Power Rule*:

1. The integrand does not match any of the basic trigonometric integration formulas!

2. The integrand contains a trigonometric expression raised to a power.

Here you have to be very careful with substitution!

a. Let's investigate what will happen if we use the following substitution:

u = sin x

du = cos dx

Since the right side of the *du*-equation matches the remaining factors ***dx*** and ***cos x*** of the integrand exactly, we are now assured that we can use the *General Power Rule for Integration* to find the family of antiderivatives.

We proceed to write the integral in terms of ***u*** and evaluate as follows:

$$\int u \, du = u^2 + C$$

and since the variable of integration is ***x***, we'll convert back to the original variable to find that

$$\int \sin x \cos x \, dx = \tfrac{1}{2}\sin^2 x + C$$

Therefore, the antiderivative is $F(x) = \frac{1}{2}\sin^2 x + C$

b. Now let's investigate what will happen if we use the substitution

u = cos x

du = - sin dx

The right side of the *du*-equation contains the remaining factors ***sin x*** and ***dx*** of the integrand, but the integrand does not have a constant factor of ***-1***. This is easily fixed by multiplying the *du*-equation as follows:

- du = sin x dx

Now, the right side of the *du*-equation matches the remaining factors of the integrand exactly. We are now assured that we can use the *General Power Rule for Integration* to find the family of antiderivatives.

We proceed to write the integral in terms of ***u*** and evaluate as follows:

$- \int u \, du = - u^2 + C$

and since the variable of integration is ***x***, we'll convert back to the original variable to find that

$$\int \sin x \cos x \, dx = -\tfrac{1}{2}\cos^2 x + C$$

Therefore, the antiderivative is $F(x) = -\frac{1}{2}\cos^2 x + C$

Comparing the antiderivatives derived from the two different methods used in (a) and (b), they don't seem to be the same.

That is,

(a) $\int \sin x \cos x \, dx = \tfrac{1}{2}\sin^2 x + C$

(b) $\int \sin x \cos x \, dx = -\tfrac{1}{2}\cos^2 x + C$

However, we can make them be the same by using the *Pythagorean Identity* $\cos^2 x = 1 - \sin^2 x$.

Then the antiderivative in (b) can be rewritten as

$$\int \sin x \cos x\,dx = -\tfrac{1}{2}\cos^2 x + C$$
$$= -\tfrac{1}{2}(1-\sin^2 x) + C$$
$$= -\tfrac{1}{2} + \tfrac{1}{2}\sin^2 x + C$$

Comparing the antiderivatives derived now, we find that one contains the constant $-\frac{1}{2}$, whereas the other one does not.

Are the two antiderivatives equivalent? Yes they are! One antiderivative simply shows more of the constant factor ***C*** than the other one. As a matter of fact we could eliminate the constant $-\frac{1}{2}$ in (b).

Problem 12:

Evaluate $\displaystyle\int \frac{\cos 2x}{\sin^5 2x}\,dx$

First let's write the integrand as a product $\int \cos 2x(\sin 2x)^{-5}\,dx$

Reasons for trying to use the *General Power Rule*:

1. The integrand does not match any of the basic trigonometric integration formulas!

2. The integrand contains a trigonometric expression raised to a power.

NOTE: Due to the reasons above, the *Chain Rule for Integration* cannot be considered as the primary integration rule!

Using *u*-substitution as follows, we find

u = sin 2x

du = 2 cos 2x dx

NOTE: Here we had to use the *Chain Rule for Differentiation*.

The right side of the *du*-equation contains the remaining factors ***cos 2x*** and ***dx*** of the integrand, but the integrand does not have a constant factor of ***2***. This is easily fixed by multiplying the *du*-equation as follows:

du = cos 2x dx

Now, the right side of the *du*-equation matches the remaining factors of the integrand exactly. We are now assured that we can use the *General Power Rule for Integration* to find the family of antiderivatives.

We proceed to write the integral in terms of ***u*** and evaluate as follows:

$$\int u^{-5}\,du = (-u^{-4}) + C$$

and since the variable of integration is ***x***, we'll convert back to the original variable to find that

$$\int \frac{\cos 2x}{\sin^5 2x}\,dx = -\tfrac{1}{8}(\sin 2x)^{-4} + C = \frac{-1}{8\sin^4 2x} + C$$

Therefore, the antiderivative is $F(x) = \dfrac{-1}{8\sin^4 2x} + C$

Problem 13:

Evaluate $\int (1-\cos\frac{t}{2})^2 \sin\frac{t}{2}\,dt$

Reasons for trying to use the *General Power Rule*:

1. The integrand does not match any of the basic trigonometric integration formulas!

2. The integrand contains a trigonometric expression raised to a power.

NOTE: Due to the reasons above, the *Chain Rule for Integration* cannot be considered as the primary integration rule!

Using *u*-substitution as follows, we find

$$u = 1 - \cos t$$

$$du = \sin t\,dt$$

NOTE: Here we had to use the *Chain Rule for Differentiation*.

The right side of the *du*-equation contains the remaining factors ***sin t*** and ***dt*** of the integrand, but the integrand does not have a constant factor of . This is easily fixed by multiplying the *du*-equation as follows:

$$2du = \sin t\,dt$$

Now, the right side of the *du*-equation matches the remaining factors of the integrand exactly. We are now assured that we can use the *General Power Rule for Integration* to find the family of antiderivatives.

We proceed to write the integral in terms of ***u*** and evaluate as follows:

$$2\int u^2\,du = 2\left(\tfrac{1}{3}u^3\right) + C$$

and since the variable of integration is ***t***, we'll convert back to the original variable to find that

$$\int (1-\cos\tfrac{t}{2})^2 \sin\tfrac{t}{2}\, dt = \tfrac{2}{3}(1-\cos\tfrac{t}{2})^3 + C$$

Therefore, the antiderivative is $F(t) = \dfrac{2}{3}\left(1-\cos\tfrac{t}{2}\right)^3 + C$

Problem 14:

Evaluate $\displaystyle\int \frac{2e^{x} - 2e^{-x}}{(e^{x}+e^{-x})^2}\,dx$

First let's first rewrite the integrand as a product

$$\int (2e^{x}-2e^{-x})(e^{x}+e^{-x})^{-2}\,dx = \int 2(e^{x}-e^{-x})(e^{x}+e^{-x})^{-2}\,dx$$

Reasons for trying to use the *General Power Rule*:

1. The integrand does not match any of the basic trigonometric integration formulas!

2. The integrand contains a trigonometric expression raised to a power.

NOTE: Due to the reasons above, the *Chain Rule for Integration* cannot be considered as the primary integration rule!

Using *u*-substitution as follows, we find

$u = e^{x} + e^{-x}$

$du = (e^{x} - e^{-x})\, dx$

NOTE: Here we had to use the *Chain Rule for Differentiation*.

The right side of the *du*-equation contains some of the remaining factors of the integrand, but does not have a constant factor of 2. This is easily fixed by multiplying the *du*-equation as follows:

$2\, du = 2\,(e^{x} - e^{-x})\, dx$

Since now the right side of the *du*-equation matches the remaining factors of the integrand exactly, we are now assured that we can use the *General Power Rule for Integration* to find the family of antiderivatives.

We proceed to write the integral in terms of u and evaluate as follows:

$2 \int u^{-2}\, du = -2\, u^{-1} + C$

and since the variable of integration is v, we'll convert back to the original variable to find that

$$\int \frac{2e^{x} - 2e^{-x}}{(e^{x} + e^{-x})^{2}}\,dx = -2(e^{x} + e^{-x})^{-1} + C = \frac{-2}{e^{x} + e^{-x}} + C$$

Therefore, the antiderivative is $F(x) = \dfrac{-2}{e^{x} + e^{-x}} + C$

Problem 15:

Evaluate $\displaystyle\int \frac{1}{4x - 1}\,dx$

First let's first write the integrand as a product

$$\int (4x - 1)^{-1}\,dx$$

Since there is a power of ***-1***, we should try the *Log Rule for Integration* to evaluate the integral.

Using *u*-substitution as follows, we find

u = 4x - 1

du = 4 dx

The right side of the *du*-equation does not quite match the remaining factors of the integrand because the constant factor of the integrand is ***1***. This is easily fixed by multiplying the *du*-equation as follows:

du = dx

Since now the right side of the *du*-equation matches the remaining factor ***dx*** of the integrand exactly, we are now assured that we can use the *Log Rule for Integration* to find the family of antiderivatives.

We proceed to write the integral in terms of ***u*** and evaluate as follows:

$\int u^{-1}\,du = \ln|u| + C$

and since the variable of integration is ***x***, we'll convert back to the original variable to find that

$$\int \frac{1}{4x - 1}\,dx = \tfrac{1}{4}\ln|4x - 1| + C$$

Therefore, the antiderivative is $F(x) = \dfrac{1}{4}\ln|4x - 1| + C$

Problem 16:

Evaluate $\int \frac{3x^2+1}{x^3+x}\,dx$

First let's first write the integrand as a product

$$\int (3x^2+1)(x^3+x)^{-1}\,dx$$

Since there is a power of ***-1***, we should try the *Log Rule for Integration* to evaluate the integral.

Using *u*-substitution as follows, we find

$u = x^3 + x$

$du = (3x^2 + 1)\,dx$

Since the right side of the *du*-equation matches the remaining factors of the integrand exactly, we are now assured that we can use the *Log Rule for Integration* to find the family of antiderivatives.

We proceed to write the integral in terms of ***u*** and evaluate as follows:

$\int u^{-1}\,du = \ln|u| + C$

and since the variable of integration is ***x***, we'll convert back to the original variable to find that

$$\int \frac{3x^2+1}{x^3+x}\,dx = \ln\left|x^3+x\right| + C$$

Therefore, the antiderivative is $F(x) = \ln\left|x^3+x\right| + C$

Problem 17:

Evaluate $\int \frac{\sec^2 x}{\tan x}\,dx$

First let's first write the integrand as a product

$$\int \sec^2 x\,(\tan x)^{-1}\,dx$$

Since there is a power of ***-1***, we should try the *Log Rule for Integration* to evaluate the integral.

Using *u*-substitution as follows, we find

$u = \tan x$

$du = \sec^2 x \, dx$

Since the right side of the *du*-equation matches the remaining factors of the integrand exactly, we are now assured that we can use the *Log Rule for Integration* to find the family of antiderivatives.

We proceed to write the integral in terms of u and evaluate as follows:

$\int u^{-1} \, du = \ln |u| + C$

and since the variable of integration is x, we'll convert back to the original variable to find that

$$\int \frac{\sec^2 x}{\tan x} \, dx = \ln |\tan x| + C$$

Therefore, the antiderivative is $F(x) = \ln |\tan x| + C$

Problem 18:

Evaluate $\int \frac{x+1}{x^2+2x} \, dx$

First let's first write the integrand as a product

$$\int (x+1)(x^2+2x)^{-1} \, dx$$

Since there is a power of -1, we should try the *Log Rule for Integration* to evaluate the integral.

Using *u*-substitution as follows, we find

$u = x^2 + 2x$

$du = 2(x + 1) \, dx$

The right side of the *du*-equation does not quite match the remaining factors of the integrand because the constant factor of the integrand is 1. This is easily fixed by multiplying the *du*-equation as follows:

$du = (x + 1) \, dx$

Since now the right side of the *du*-equation matches the remaining factors dx and $(x + 1)$ of the integrand exactly, we are now assured that we can use the *Log Rule for Integration* to find the family of antiderivatives.

We proceed to write the integral in terms of u and evaluate as follows:

$$\int u^{-1}\,du = \ln|u| + C$$

and since the variable of integration is x, we'll convert back to the original variable to find that

$$\int \frac{x+1}{x^2+2x}\,dx = \tfrac{1}{2}\ln\left|x^2+2x\right| + C$$

Therefore, the antiderivative is $F(x) = \dfrac{1}{2}\ln\left|x^2+2x\right| + C$

Problem 19:

Evaluate $\displaystyle\int \frac{x^2+x+1}{x^2+1}\,dx$

First let's first write the integrand as a product

$$\int (x^2+x+1)(x^2+1)^{-1}\,dx$$

Since there is a power of -1, we should try the *Log Rule for Integration* to evaluate the integral.

Using *u*-substitution as follows, we find

$$u = x^2 + 1$$

$$du = 2x\,dx$$

The right side of the *du*-equation does not match the remaining factors of the integrand. First of all the constant factor of the integrand is 1. Secondly, the integrand has a factor of $(x^2 + x + 1)$ and the *du*-equation has a factor of x. Taking care of the constant factor isn't a problem as we have seen in earlier examples. However, there is NOTHING that can be done concerning the differing factors !!!

There is one thing we can try before we give up and that is to use long division as follows to reduce the integrand.

$$\begin{array}{r|l} & \quad\quad\quad\quad 1 \\ x^2+1 & x^2+x+1 \\ & -(x^2 \quad\quad +1) \\ \hline & \quad\quad x \end{array}$$

Now we can write the integrand as follows:

$$\int\left(1+\frac{x}{x^2+1}\right)dx = \int dx + \int \frac{x}{x^2+1}\,dx$$

Integration of the first integral is not a problem. However, let's write the second integral as a product to see if anything further can be done.

That is, $\int x(x^2+1)^{-1}\,dx$

Since there is a power of ***-1***, we should try the *Log Rule for Integration* again to evaluate the integral.

Using *u*-substitution as follows, we find

$u = x^2 + 1$

$du = 2x\,dx$

The right side of the *du*-equation does not quite match the remaining factors of the integrand because the constant factor of the integrand is ***1***. This is easily fixed by multiplying the *du*-equation as follows:

$du = x\,dx$

Since now the right side of the *du*-equation matches the remaining factors ***x*** and ***dx*** of the integrand exactly, we are now assured that we can use the *Log Rule for Integration* to find the family of antiderivatives.

We proceed to write the integral in terms of ***u*** and evaluate as follows:

$\int u^{-1}\,du = \ln|u| + C$

and since the variable of integration is ***x***, we'll convert back to the original variable to find that

$$\int x(x^2+1)^{-1}\,dx = \tfrac{1}{2}\ln\left|x^2+1\right| + C$$

Therefore, the antiderivative of our original integral is as follows:

$$\begin{aligned}\int \frac{x^2+x+1}{x^2+1}\,dx &= \int\left(1+\frac{x}{x^2+1}\right)dx\\ &= \int dx + \int \frac{x}{x^2+1}\,dx\\ &= x + \tfrac{1}{2}\ln\left|x^2+1\right| + C\end{aligned}$$

Therefore, the antiderivative is $F(x) = x + \dfrac{1}{2}\ln\left|x^2+1\right| + C$

Problem 20:

Evaluate $\int \tan x \; dx$

This integral does not seem to fit any formulas on our basic list of trigonometric integrals. However, by using a trigonometric identity, you can obtain the following.

$$\int \tan x \; dx = \int \frac{\sin x}{\cos x} dx$$

Now, let's write the integrand as a product

$$\int \sin x (\cos x)^{-1} \; dx$$

Since there is a power of ***-1***, we should try the *Log Rule for Integration* to evaluate the integral.

Using *u*-substitution as follows, we find

u = cos x

du = - sin x dx

The right side of the *du*-equation does not quite match the remaining factors of the integrand because the constant factor of the integrand is ***1***. This is easily fixed by multiplying the *du*-equation as follows:

- du = sin x dx

Since now the right side of the *du*-equation matches the remaining factors ***dx*** and ***sin x*** of the integrand exactly, we are now assured that we can use the *Log Rule for Integration* to find the family of antiderivatives.

We proceed to write the integral in terms of ***u*** and evaluate as follows:

$$-\int u^{-1} \, du = -\ln|u| + C$$

and since the variable of integration is ***x***, we'll convert back to the original variable to find that

$$\int \tan x \; dx = -\ln|\cos x| + C$$

Therefore, the antiderivative is $F(x) = -\ln|\cos x| + C$

Problem 21:

Evaluate $\int \sec x \ dx$

This integral also does not seem to fit any formulas on our basic list of trigonometric integrals. However, by using a "trick", you can obtain the following.

$$\int \sec x \ dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx$$
$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

Please note that we multiplied the integrand by a special form of the number 1 !!!

Now, let's write the integrand as a product

$$\int (\sec^2 x + \sec x \tan x)(\sec x + \tan x)^{-1} \ dx$$

Since there is a power of ***-1***, we should try the *Log Rule for Integration* to evaluate the integral.

Using *u*-substitution as follows, we find

$u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x) \ dx$

Since now the right side of the *du*-equation matches the remaining factors of the integrand exactly, we are now assured that we can use the *Log Rule for Integration* to find the family of antiderivatives.

We proceed to write the integral in terms of ***u*** and evaluate as follows:

$\int u^{-1} \ du = \ln |u| + C$

and since the variable of integration is ***x***, we'll convert back to the original variable to find that

$$\int \sec x \ dx = \ln |\sec x + \tan x| + C$$

Therefore, the antiderivative is $F(x) = -\ln |\sec x + \tan x| + C$

Problem 22:

Evaluate $\int \frac{e^{2x}}{1+e^{2x}}\,dx$

First let's first write the integrand as a product

$$\int e^{2x}(1+e^{2x})^{-1}\,dx$$

Since there is a power of ***-1***, we should try the *Log Rule for Integration* to evaluate the integral.

Using *u*-substitution as follows, we find

$u = 1 + e^{2x}$

$du = e^{2x}\,dx$

Since the right side of the *du*-equation matches the remaining factors of the integrand exactly, we are now assured that we can use the *Log Rule for Integration* to find the family of antiderivatives.

We proceed to write the integral in terms of ***u*** and evaluate as follows:

$\int u^{-1}\,du = \ln|u| + C$

and since the variable of integration is ***x***, we'll convert back to the original variable to find that

$$\int \frac{e^{2x}}{1+e^{2x}}\,dx = \ln\left|1+e^{2x}\right| + C$$

Therefore, the antiderivative is $F(x) = \ln\left|1+e^{2x}\right| + C$